

F/G 12/1

F/G 12/1

DAAG29-75-C-0024  
NL

NL



84  
 85  
 86  
 87  
 88  
 89  
 90  
 91  
 92  
 93  
 94  
 95  
 96  
 97  
 98  
 99  
 100  
 101  
 102  
 103  
 104  
 105  
 106  
 107  
 108  
 109  
 110  
 111  
 112  
 113  
 114  
 115  
 116  
 117  
 118  
 119  
 120  
 121  
 122  
 123  
 124  
 125  
 126  
 127  
 128  
 129  
 130  
 131  
 132  
 133  
 134  
 135  
 136  
 137  
 138  
 139  
 140  
 141  
 142  
 143  
 144  
 145  
 146  
 147  
 148  
 149  
 150  
 151  
 152  
 153  
 154  
 155  
 156  
 157  
 158  
 159  
 160  
 161  
 162  
 163  
 164  
 165  
 166  
 167  
 168  
 169  
 170  
 171  
 172  
 173  
 174  
 175  
 176  
 177  
 178  
 179  
 180  
 181  
 182  
 183  
 184  
 185  
 186  
 187  
 188  
 189  
 190  
 191  
 192  
 193  
 194  
 195  
 196  
 197  
 198  
 199  
 200  
 201  
 202  
 203  
 204  
 205  
 206  
 207  
 208  
 209  
 210  
 211  
 212  
 213  
 214  
 215  
 216  
 217  
 218  
 219  
 220  
 221  
 222  
 223  
 224  
 225  
 226  
 227  
 228  
 229  
 230  
 231  
 232  
 233  
 234  
 235  
 236  
 237  
 238  
 239  
 240  
 241  
 242  
 243  
 244  
 245  
 246  
 247  
 248  
 249  
 250  
 251  
 252  
 253  
 254  
 255  
 256  
 257  
 258  
 259  
 260  
 261  
 262  
 263  
 264  
 265  
 266  
 267  
 268  
 269  
 270  
 271  
 272  
 273  
 274  
 275  
 276  
 277  
 278  
 279  
 280  
 281  
 282  
 283  
 284  
 285  
 286  
 287  
 288  
 289  
 290  
 291  
 292  
 293  
 294  
 295  
 296  
 297  
 298  
 299  
 300  
 301  
 302  
 303  
 304  
 305  
 306  
 307  
 308  
 309  
 310  
 311  
 312  
 313  
 314  
 315  
 316  
 317  
 318  
 319  
 320  
 321  
 322  
 323  
 324  
 325  
 326  
 327  
 328  
 329  
 330  
 331  
 332  
 333  
 334  
 335  
 336  
 337  
 338  
 339  
 340  
 341  
 342  
 343  
 344  
 345  
 346  
 347  
 348  
 349  
 350  
 351  
 352  
 353  
 354  
 355  
 356  
 357  
 358  
 359  
 360  
 361  
 362  
 363  
 364  
 365  
 366  
 367  
 368  
 369  
 370  
 371  
 372  
 373  
 374  
 375  
 376  
 377  
 378  
 379  
 380  
 381  
 382  
 383  
 384  
 385  
 386  
 387  
 388  
 389  
 390  
 391  
 392  
 393  
 394  
 395  
 396  
 397  
 398  
 399  
 400  
 401  
 402  
 403  
 404  
 405  
 406  
 407  
 408  
 409  
 410  
 411  
 412  
 413  
 414  
 415  
 416  
 417  
 418  
 419  
 420  
 421  
 422  
 423  
 424  
 425  
 426  
 427  
 428  
 429  
 430  
 431  
 432  
 433  
 434  
 435  
 436  
 437  
 438  
 439  
 440  
 441  
 442  
 443  
 444  
 445  
 446  
 447  
 448  
 449  
 450  
 451  
 452  
 453  
 454  
 455  
 456  
 457  
 458  
 459  
 460  
 461  
 462  
 463  
 464  
 465  
 466  
 467  
 468  
 469  
 470  
 471  
 472  
 473  
 474  
 475  
 476  
 477  
 478  
 479  
 480  
 481  
 482  
 483  
 484  
 485  
 486  
 487  
 488  
 489  
 490  
 491  
 492  
 493  
 494  
 495  
 496  
 497  
 498  
 499  
 500  
 501  
 502  
 503  
 504  
 505  
 506  
 507  
 508  
 509  
 510  
 511  
 512  
 513  
 514  
 515  
 516  
 517  
 518  
 519  
 520  
 521  
 522  
 523  
 524  
 525  
 526  
 527  
 528  
 529  
 530  
 531  
 532  
 533  
 534  
 535  
 536  
 537  
 538  
 539  
 540  
 541  
 542  
 543  
 544  
 545  
 546  
 547  
 548  
 549  
 550  
 551  
 552  
 553  
 554  
 555  
 556  
 557  
 558  
 559  
 560  
 561  
 562  
 563  
 564  
 565  
 566  
 567  
 568  
 569  
 570  
 571  
 572  
 573  
 574  
 575  
 576  
 577  
 578  
 579  
 580  
 581  
 582  
 583  
 584  
 585  
 586  
 587  
 588  
 589  
 590  
 591  
 592  
 593  
 594  
 595  
 596  
 597

AD A 038959

MRC Technical Summary Report #1726

ORTHOGONAL POLYNOMIALS

Paul G. Nevai

12

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

February 1977

(Received December 9, 1976)



*See form 1473*

Approved for public release  
Distribution unlimited

AD No.  
DDC FILE COPY

Sponsored by  
U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

American Mathematical Society  
Providence, Rhode Island 02940  
National Science Foundation  
Washington, D. C. 20550



UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

ORTHOGONAL POLYNOMIALS

Paul G. Nevai

Technical Summary Report #1726  
February 1977

ABSTRACT

The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegő and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem will (positively) be answered: is there any weight  $w$  with compact support such that for each  $p > 2$  the Lagrange interpolating polynomials corresponding to  $w$  diverge in  $L_w^p$  for some continuous function  $f$ ? Most of the paper deals with Christoffel functions and their applications. Many limit relations for orthogonal polynomials are found in the assumption that the coefficients in the recursion formula behave nicely.

AMS(MOS) Subject Classification - 42A52, 33A65, 41A05, 41A10, 41A20, 41A25, 41A35, 41A55, 41A60, 42A04, 42A56, 42A62, 40A25, 26A75, 26A78, 26A82, 26A84, 30A06, 30A84, 30A86, 65D30, 65F15, 65F35.

Key Words - Orthogonal polynomials, Quadrature processes, Fourier series, Interpolation, Positive operators, Toeplitz matrices, Christoffel functions.

Work Unit Number 6 - Spline Functions and Approximation Theory

---

Sponsored by

- 1) the United States Army under Contract No. DAAG29-75-C-0024,
- 2) the American Mathematical Society, Providence, Rhode Island 02940,
- 3) the National Science Foundation, under Grant No. MPS75-06687 #3.

This work is dedicated

to

Richard Askey

ACCESSION FOR	
NTIS	White Section <input checked="" type="checkbox"/>
DDC	Buff Section <input type="checkbox"/>
UNANNOUNCED	
JUSTIFICATION	
BY	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. and/or SPECIAL
A	

## Table of Contents

1.	Introduction . . . . .	1
2.	Notations . . . . .	4
3.	Basic Facts . . . . .	10
	3.1. The Generalized Recurrence Formula . . . . .	10
	3.2. Modified Quadrature Processes . . . . .	20
	3.3. The Support of $d\alpha$ . . . . .	25
4.	Limit Relations . . . . .	32
	4.1. Pointwise Limits . . . . .	32
	4.2. Weak Limits . . . . .	50
5.	Eigenvalues of Toeplitz Matrices . . . . .	63
6.	Christoffel Functions . . . . .	73
	6.1. An Interpolation Process. . . . .	73
	6.2. A Sequence of Positive Operators . . . . .	98
	6.3. Generalized Christoffel Functions. . . . .	140
7.	The Coefficients in the Recurrence Formula. . . . .	163
8.	Fourier Series . . . . .	188
9.	Inequalities . . . . .	198
10.	Lagrange Interpolation . . . . .	220
	References. . . . .	230

# ORTHOGONAL POLYNOMIALS

Paul G. Nevai

The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegő and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem [1] will be answered: is there any weight  $w$  with compact support such that for each  $p > 2$  the Lagrange interpolating polynomials corresponding to  $w$  diverge in  $L^p_w$  for some continuous function  $f$ ? R. Askey [1] conjectured that the answer was yes and the solution was given by the Pollaczek weight because the logarithm of the Pollaczek weight is not integrable. We shall show that Askey's conjecture is right but for different reasons. In fact, there are many weights solving Turan's problem; some of them do have integrable logarithm, some of them do not.

Most of this paper deals with investigation of Christoffel functions and its generalization. The results and the methods are stronger than those of the above authors. The Christoffel functions play a very important role in the theory of orthogonal polynomials. Many results in orthogonal Fourier series and interpolation are based on estimates and asymptotics of Christoffel functions. We shall show how successfully Christoffel functions can be applied in finding necessary conditions for weighted mean convergence of orthogonal Fourier series and Lagrange

Sponsored by

- 1) The United States Army under Contract No. DAAG29-75-C-0024,
- 2) The American Mathematical Society, Providence, Rhode Island 02940,
- 3) The National Science Foundation, under Grant No. MPS75-06687 #3.

interpolation processes. Introducing generalized Christoffel functions we shall find a connection between different weighted  $L^p$  norms of polynomials. Especially interesting is the case when  $0 < p < 1$ . We shall investigate a new kind of quadrature process which helps to find asymptotics for Christoffel functions and orthogonal polynomials outside the support of the weight function. Taking the recursion formula as a starting point and assuming properties on the coefficients in the recursion formula we shall obtain results on orthogonal polynomials. In certain cases we shall be able to calculate the weight function using the coefficients in the recursion formula. We shall also find the  $(C, 1)$  limit of Turan type determinants under rather weak conditions.

I learned the theory of orthogonal polynomials from G. Freud who has been supervising me for several years. Many of the methods I use in this paper can be found in his book on orthogonal polynomials which is a rich source of methods and unsolved research problems. I wish to express my deep feeling of gratitude to G. Freud as well as to R. Askey, L. Bers, M. Cwikel, J. Landin, G. G. Lorentz and W. Proxmire without the help of which this paper would never have been written. I am grateful to the American Mathematical Society, to the National Science Foundation and to the United States Army for sponsoring my research.

This whole work was born from the attempts to solve problems mentioned in R. Askey's paper [1]. I discussed my results with R. Askey several times. He read a draft version of the manuscript and made various



suggestions to improve the presentation. I dedicate this work to Richard Askey because his generous support and help made it possible for me to carry out the research which led to this paper.

## 2. Notations

The function  $\alpha: \mathbb{R} \rightarrow \mathbb{R}$  is called a weight function if it is nondecreasing, it has infinitely many points of increase and all the moments

$$\int_{-\infty}^{\infty} x^{2n} d\alpha(x) \quad (n = 0, 1, \dots)$$

are finite. For a given weight  $\alpha$  the corresponding system of orthogonal polynomials  $\{p_n(d\alpha)\}_{n=0}^{\infty}$  is defined by  $p_n(d\alpha, x) = Y_n(d\alpha)x^n + \dots, Y_n(d\alpha) > 0$  and

$$\int_{-\infty}^{\infty} p_n(d\alpha, x) p_m(d\alpha, x) d\alpha(x) = \delta_{nm}.$$

If  $\alpha$  happens to be absolutely continuous then we shall usually write  $w$  and  $p_n(w, x)$  instead of  $\alpha'$  and  $p_n(d\alpha, x)$  respectively. In the general case  $\alpha$  can be written in the form

$$\alpha = \alpha_{ac} + \alpha_s + \alpha_j$$

where  $\alpha_{ac}$  is absolutely continuous,  $\alpha_s$  is singular and  $\alpha_j$  is a jump function.

One of the basic properties of a system of orthogonal polynomials

$\{p_n(d\alpha)\}$  is that the polynomials  $p_n(d\alpha)$  satisfy the three term recurrence relation

$$\begin{aligned} x p_n(d\alpha, x) &= \frac{Y_n(d\alpha)}{Y_{n+1}(d\alpha)} p_{n+1}(d\alpha, x) + \\ &+ \alpha_n(d\alpha) p_n(d\alpha, x) + \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} p_{n-1}(d\alpha, x) \end{aligned}$$

$(n = 0, 1, \dots)$  where  $p_{-1} \equiv 0$  and

$$\alpha_n(d\alpha) = \int_{-\infty}^{\infty} t^2 p_n(d\alpha, t) d\alpha(t).$$

This recurrence relation will be one of our main points of interest. By a famous result of J. Favard if a system of polynomials  $\{p_n(x)\}_{n=0}^{\infty}$  satisfies the recurrence formula

$$x p_n(x) = \frac{y_n}{y_{n+1}} p_{n+1}(x) + \alpha_n \frac{p_n(x)}{y_n} + \frac{y_{n-1}}{y_n} p_{n-1}(x)$$

for  $n = 0, 1, \dots$  with  $p_{-1} \equiv 0$ ,  $p_0 \equiv y_0$ ,  $y_n > 0$  and  $\alpha_n \in \mathbb{R}$  then  $\{p_n(x)\}$  is orthogonal with respect to some weight  $\alpha$  which may not uniquely be determined (see Freud, §II.1)\*. In this paper we are going to deal with such cases when both  $\{\alpha_n\}$  and  $\{y_{n-1}/y_n\}$  are bounded and then  $\alpha$  is uniquely defined.

The zeros of  $p_n(d\alpha)$ , which are real and distinct, will be denoted by  $x_{kn}(d\alpha)$ :  $x_{1n}(d\alpha) > x_{2n}(d\alpha) > \dots > x_{nn}(d\alpha)$ . The Christoffel function  $\lambda_n(d\alpha)$  corresponding to  $\alpha$  is defined by

$$\lambda_n(d\alpha, z) = \min_{\pi \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t)$$

$$\pi(t) \equiv 1$$

for  $z \in C$ ,  $n = 1, 2, \dots$  where  $\mathcal{P}_n$  is the set of polynomials of degree at most  $n$ . It is rather easy to see that

$$\lambda_n(d\alpha, z)^{-1} = \sum_{k=0}^{n-1} |p_k(d\alpha, z)|^2$$

The numbers  $\lambda_n(d\alpha, x_{kn}(d\alpha))$  are called Christoffel numbers and are usually

\* In the following books listed in the references will be referred by mentioning the name of their authors.

denoted by  $\lambda_{kn}(d\alpha)$ . There are two important results involving Christoffel numbers which will often be used. The first of them is the Gauss-Jacobi mechanical quadrature formula:

$$\int_{-\infty}^{\infty} \pi(t) d\alpha(t) = \sum_{k=1}^n \lambda_{kn}(d\alpha) \pi(x_{kn}(d\alpha))$$

for each  $\pi \in \mathcal{P}_{2n-1}$ , the second one is the Markov-Stieltjes inequalities which can be expressed as

$$\sum_{k=1}^n \lambda_{kn}(d\alpha) \leq \int_{-\infty}^{\infty} d\alpha(t) \leq \sum_{k=1}^n \lambda_{kn}(d\alpha)$$

( $i = 1, 2, \dots, n$ ). The support of  $\alpha$ , that is  $\text{supp}(d\alpha)$ , is the set of points of increase of  $\alpha$ . If  $\text{supp}(d\alpha)$  is bounded  $\Delta(d\alpha)$  will denote the smallest closed interval containing  $\text{supp}(d\alpha)$ . The symbols  $\Delta$  and  $\tau$  will always mean closed interval, the interior part of  $\Delta$  is denoted by  $\Delta^0$ . For a given  $\tau$  the Tschebyshev weight corresponding to  $\tau$  will be written as  $v_{\tau}$ . If  $\tau = [-1, 1]$  then we write  $v$  instead of  $v_{\tau}$ . If  $\tau = [a-b, a+b]$  then

$$v_{\tau}(x) = [b^2 - (x-a)^2]^{-\frac{1}{2}}.$$

For  $f \in L^1_{d\alpha}$  ( $\text{supp}(d\alpha)$  is bounded)  $S_n(d\alpha, f)$  denotes the  $n$ -th partial sum of the orthogonal Fourier series of  $f$ . Hence for  $x \in \mathbb{R}$

$$S_n(d\alpha, f, x) = \int_{-\infty}^{\infty} f(t) K_n(d\alpha, x, t) d\alpha(t)$$

where



$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} P_k(d\alpha, x) P_k(d\alpha, t)$$

or by the Christoffel-Darboux formula

$$K_n(d\alpha, x, t) = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \frac{P_{n-1}(d\alpha, t) P_n(d\alpha, x) - P_{n-1}(d\alpha, x) P_n(d\alpha, t)}{x-t}.$$

For a given function  $f$  the Lagrange interpolation polynomial  $L_n(d\alpha, f)$  corresponding to  $\alpha$  is defined to be the unique polynomial of degree at most  $n-1$  which agrees with  $f$  at the nodes  $x_{kn}(d\alpha)$  ( $k = 1, 2, \dots, n$ ).

If we denote by  $f_{kn}(d\alpha)$  the fundamental polynomials of Lagrange interpolation then  $L_n(d\alpha, f)$  can be written as

$$L_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}(d\alpha)) f_{kn}(d\alpha, x).$$

It will be useful to remember that

$$f_{kn}(d\alpha, x) = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \lambda_{kn}(d\alpha) P_{n-1}(d\alpha, x_{kn}) \frac{P_n(d\alpha, x)}{x - x_{kn}}$$

$$(x_{kn} \equiv x_{kn}(d\alpha)).$$

The Tschebyshev polynomials  $\cos n\theta$  ( $x = \cos \theta$ ) will always be denoted by  $T_n(x)$ . For a given set  $\mathbb{M}$  the characteristic function of  $\mathbb{M}$  is  $1_{\mathbb{M}}$  and  $\mathbb{M}(\varepsilon)$  ( $\varepsilon > 0$ ) means the  $\varepsilon$ -neighborhood of  $\mathbb{M}$ .  $\pi_n$  and  $P_n$  denote polynomials belonging to  $\mathbb{P}_n$ . The letters  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  denote the set of natural integers, real numbers and complex numbers respectively.  $\mathbb{R}^+$  is the set of positive real numbers.

For  $0 < p < \infty$   $\| \cdot \|_{d\alpha, p}$  is defined by

$$\|f\|_{d\alpha, p}^p = \int_{-\infty}^{\infty} |f(t)|^p d\alpha(t).$$

(Of course, for  $0 < p < 1$  this is not a norm.)

Sometimes we shall omit unnecessary parameters in the formulas.

(E.g.  $x_k \equiv x_{kn}(d\alpha)$ ).

We assume that the reader is familiar with methods in the theory of one-sided approximation and positive operators.

For the convenience of the reader we give an index where to find the definition of symbols used frequently. D. 3.1.4 below means that see Definition 4 in Chapter 3.1.

$a_n$	- D. 7.6
$A_x^\omega, A_x^\omega, B_x^\omega, B_x^\omega$	- D. 6.2.37
$\alpha_g$	- D. 6.1.3
$\alpha_r, \beta_r$	- D. 6.2.42
$\alpha_{nk}(d\alpha)$	- D. 4.2.11
$C_k^{a,b}(d\alpha)$	- D. 3.1.4
$D(d\alpha, z)$	- D. 6.1.16
$\delta_t$	- D. 7.14
$GJ$	- D. 9.28
$\Gamma(\theta)$	- D. 4.2.4
$JS$	- D. 10.17
$\lambda_n(d\alpha, p, x)$	- D. 6.3.1
$\lambda_n^*(d\alpha, x)$	- Formula 4.1(5)

M(a, b)	- D. 3. 1. 6
Pollaczek weight	- D. 6. 2. 12
$\rho(z)$	- D. 4. 1. 8
S	- D. 4. 2. 1
$u \equiv u(a, b)$	- D. 6. 2. 7
$u_n$	- D. 6. 3. 4
$w_n$	- D. 9. 28
$\sim$	- D. 6. 3. 3

### 3. Basic Facts

#### 3.1. The Generalized Recurrence Formula

Let  $U_n(x)$  denote the Tschhebychev polynomial of second kind, that is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = \cos \theta$$

( $0 \leq \theta \leq \pi$ ,  $-1 \leq x \leq 1$ ). The polynomials  $U_n(x)$  satisfy the recurrence formula

$$(1) \quad 2x U_{n-1}(x) = U_n(x) + U_{n-2}(x), \quad n = 1, 2, \dots$$

where  $U_{-1}(x) \equiv 0$  and  $U_0(x) \equiv 1$ .

Theorem 1. Let  $0 \leq k \leq n$ . For an arbitrary weight  $\alpha$   $p_n(d\alpha, x)$  can be expressed as

$$(2) \quad p_n(d\alpha, x) = U_{n-k}(x) p_k(d\alpha, x) - U_{n-k-1}(x) p_{k-1}(d\alpha, x) + R_{n,k}(d\alpha, x)$$

where

$$(3) \quad R_{n,k}(d\alpha, x) = \sum_{j=k+1}^n U_{n-j}(x) \left\{ \left[ 1 - 2 \frac{y_{j-1}(d\alpha)}{y_{j-1}(d\alpha)} \right] p_j(d\alpha, x) - \left[ 1 - 2 \frac{y_{j-2}(d\alpha)}{y_{j-1}(d\alpha)} \right] p_{j-2}(d\alpha, x) \right\}.$$

Proof. We shall prove (2) by induction. If  $k = n$  then (2) and (3) give  $p_n(d\alpha, x) = p_n(d\alpha, x)$ . If  $n \geq 1$  and  $k+1 = n$  then (2) and (3) coincide with the recurrence formula. Now fix  $n$  and let  $n-1 > k \geq 0$ . Suppose

that (2) and (3) hold if we replace these  $k$  by  $k+1$ , that is

$$p_n = U_{n-k-1} p_{k+1} - U_{n-k-2} p_k + R_{n,k+1}.$$

Applying (2) and (3) to the case  $n = k+1 \geq 1$  we obtain

$$p_{k+1} = U_1 p_k - U_0 p_{k-1} + R_{k+1,k}.$$

Thus by (i)

$$\begin{aligned} p_n &= (U_1 U_{n-k-1} - U_0 U_{n-k-2}) p_k - U_{n-k-1} p_{k-1} + \\ &+ R_{n,k+1} + U_{n-k-1} R_{k+1,k} = \\ &= U_{n-k} p_k - U_{n-k-1} p_{k-1} + R_{n,k}, \end{aligned}$$

that is (2) and (3) hold also for  $k$ .

Remark 2. Putting  $k = 0$  and  $d\alpha = \text{Tschebyshev weight}$  we obtain from Theorem 1

$$T_n(x) = x U_{n-1}(x) - U_{n-2}(x) = T_1(x) U_{n-1}(x) - T_0(x) U_{n-2}(x).$$

For  $k = 1$  (i) and (3) give the same as for  $k = 0$ . When  $2 \leq k \leq n$  we get

$$T_n(x) = T_k(x) U_{n-k}(x) - T_{k-1}(x) U_{n-k-1}(x).$$

This formula may easily be checked directly. In fact, the above formula suggested (2) and (3).

Theorem 3. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ ,  $0 \leq k \leq n$ . Then

-11-

$$\begin{aligned} (4) \quad p_n(d\alpha, x) &= U_{n-k} \left( \frac{x-a}{b} \right) p_k(d\alpha, x) - \\ &- U_{n-k-1} \left( \frac{x-a}{b} \right) p_{k-1}(d\alpha, x) + R_{n,k}^{a,b}(d\alpha, x) \end{aligned}$$

where

$$\begin{aligned} (5) \quad R_{n,k}^{a,b}(d\alpha, x) &= \sum_{j=k+1}^n U_{n-j} \left( \frac{x-a}{b} \right) \cdot \\ &\cdot \left\{ \left[ 1 - \frac{2}{b} \frac{Y_{j-1}(d\alpha)}{Y_j(d\alpha)} \right] p_j(d\alpha, x) + \frac{2}{b} [a - \sigma_{j-1}(d\alpha)] p_{j-1}(d\alpha, x) + \right. \\ &\quad \left. + \left[ 1 - \frac{2}{b} \frac{Y_{j-2}(d\alpha)}{Y_{j-1}(d\alpha)} \right] p_{j-2}(d\alpha, x) \right\}. \end{aligned}$$

Proof. Let  $\alpha^*$  be defined by  $\alpha^*(t) = \alpha(bt + a)$ . Then  $p_n(d\alpha, x) = p_n(d\alpha^*, \frac{x-a}{b})$ ,  $\alpha^*(d\alpha^*) = \frac{1}{b} [\alpha(d\alpha) - a]$  and  $Y_{n-1}(d\alpha^*)/Y_n(d\alpha^*) = \frac{1}{b} [Y_{n-1}(d\alpha)/Y_n(d\alpha)]$ . Apply now Theorem 1 to  $\alpha^*$  and then return to  $\alpha$ .

We shall call (4) and (5) the generalized recurrence formula. It will help us to prove many properties of orthogonal polynomials in case the coefficients of the recurrence formula are convergent.

Definition 4. Let  $a \in \mathbb{R}$ ,  $b \geq 0$ . Then

$$C_k^{a,b}(d\alpha) = |\alpha_k(d\alpha) - a| + \left| \frac{Y_{k-1}(d\alpha)}{Y_k(d\alpha)} - \frac{b}{2} \right| + \left| \frac{Y_k(d\alpha)}{Y_{k+1}(d\alpha)} - \frac{b}{2} \right|.$$

Corollary 5. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ ,  $0 \leq k \leq n$ . Then

$$\begin{aligned} (6) \quad p_n(d\alpha, x) &= U_{n-k} \left( \frac{x-a}{b} \right) p_k(d\alpha, x) - U_{n-k-1} \left( \frac{x-a}{b} \right) p_{k-1}(d\alpha, x) + \\ &+ O(1) \frac{1}{\sqrt{b^2 - (x-a)^2}} \sum_{j=k-1}^n C_j^{a,b}(d\alpha) |p_j(d\alpha, x)| \end{aligned}$$

-12-

for  $x \in (a-b, a+b)$  where  $|O(t)| \leq 2$ .

Definition 6. Let  $a \in \mathbb{R}$ ,  $b \geq 0$ . Then  $\alpha \in M(a, b)$  if

$$\lim_{k \rightarrow \infty} C_k^{a,b}(d\alpha) = 0.$$

Remark 7. When considering  $M(a, b)$  we can always assume without

loss of generality that either  $a = 0$ ,  $b = 1$  or  $a = 0$ ,  $b = 0$ . For, if

$\alpha \in M(a, b)$  with  $b > 0$  then  $\alpha^* \in M(0, 1)$  where  $\alpha^*(t) = \alpha(bt + a)$ . If

$\alpha \in M(a, 0)$  then  $\alpha^{**} \in M(0, 0)$  where  $\alpha^{**}(t) = \alpha(t + a)$ .

Theorem 8. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ ,  $0 < \varepsilon < 1$ ,  $x \in [a-b, a+b]$ . Then

$$\begin{aligned} [b^2 - (x-a)^2] \lambda_{n+1}(d\alpha, x) p_n^2(d\alpha, x) &\leq \\ &\leq 6 \left\{ \frac{1}{\varepsilon n} + 2 \sum_{n(1-\varepsilon) < j \leq n} [C_j^{a,b}(d\alpha)]^2 \right\} \end{aligned}$$

for  $n = 1, 2, \dots$

Proof. We obtain from (6) that for  $[(1-\varepsilon)n] + 1 \leq k \leq n$

$$\begin{aligned} [b^2 - (x-a)^2] \frac{1}{2} |p_n(x)| &\leq |p_k(x)| + |p_{k-1}(x)| + \\ &+ 2 \left\{ \sum_{j=[(1-\varepsilon)n]}^n (C_j)^2 \lambda_{n+1}^{-1}(x) \right\}^{\frac{1}{2}}, \end{aligned}$$

that is

$$[b^2 - (x-a)^2] p_n^2(x) \leq 3 p_k^2(x) + 3 p_{k-1}^2(x) + 12 \lambda_{n+1}^{-1}(x) \sum_{j=[(1-\varepsilon)n]}^n (C_j)^2.$$

Thus

$$\begin{aligned} \sum_{k=[(1-\varepsilon)n]+1}^n [b^2 - (x-a)^2] p_n^2(x) &\leq 6 \lambda_{n+1}^{-1}(x) + 12 \lambda_{n+1}^{-1}(x) \cdot \\ &\cdot \sum_{j=[(1-\varepsilon)n]}^n (C_j)^2 \sum_{k=[(1-\varepsilon)n]+1}^n 1. \end{aligned}$$

The theorem follows from this inequality.

Theorem 9. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then

$$\lim_{n \rightarrow \infty} [b^2 - (x-a)^2] \lambda_{n+1}(d\alpha, x) p_n^2(d\alpha, x) = 0$$

uniformly for  $x \in [a-b, a+b]$ .

Proof. By Theorem 8 we have to show that we can choose  $\varepsilon = \varepsilon_n \in (0, 1)$  so that

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{n\varepsilon_n} + \sum_{(1-\varepsilon_n)n \leq j \leq n} [C_j^{a,b}(d\alpha)]^2 \right\} = 0.$$

Let

$$\varepsilon_n = \frac{1}{n} \left\{ \sup_{j \geq \frac{n}{2}} [(C_j)^2 + j^{-2}] \right\}^{\frac{1}{2}}.$$

Then  $\varepsilon_n \leq \frac{1}{2}$ . Thus

$$\begin{aligned} \frac{1}{n\varepsilon_n} + \sum_{(1-\varepsilon_n)n \leq j \leq n} [C_j]^2 &\leq \frac{1}{n\varepsilon_n} + (\varepsilon_n n+1) \sup_{j \geq \frac{n}{2}} [C_j]^2 \leq \\ &\leq 2 \sup_{j \geq \frac{n}{2}} \{ [C_j]^2 + j^{-2} \}^{\frac{1}{2}} + \sup_{j \geq \frac{n}{2}} [C_j]^2 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$



Remark 10. It is useful to remember that if  $\text{supp}(d\alpha) = [-1, 1]$  and  $v \log \alpha' \in L^1$  then  $\alpha \in M(0, 1)$  (See e. g. Freud).

It might be interesting to compare Theorem 9 with (weaker) results of Geronimus (See Chapter III in his book.)

Theorem 11. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Suppose that

$$\limsup_{n \rightarrow \infty} \sum_{j=n}^{2n} |C_j^{a,b}(d\alpha)|^2 < \infty.$$

Let  $\mathfrak{M} = \overline{\mathfrak{M}} \subset (a-b, a+b)$ . Then the following two statements are equivalent.

- (i)  $\{p_n^2(d\alpha, x)\}$  is uniformly bounded for  $x \in \mathfrak{M}$ .
- (ii)  $\{n^{-1} \lambda_{n+1}^{-1}(d\alpha, x)\}$  is uniformly bounded for  $x \in \mathfrak{M}$ .

Proof. (i)  $\Rightarrow$  (ii):  $\{(n+1)^{-1} \lambda_{n+1}^{-1}(d\alpha, x)\}$  is the arithmetical mean of  $\{p_n^2(d\alpha, x)\}$ .

(ii)  $\Rightarrow$  (i): Use Theorem 8.

Let us remark that if  $C_j^{(a,b)}(d\alpha) = O(j^{-1})$  then the conditions of the theorem are satisfied. Example: Jacobi, Pollaczek polynomials.

Theorem 12. Let  $\alpha \in M(a, b)$  with  $b > 0$  and let

$$\sum_{j=0}^{\infty} C_j^{a,b}(d\alpha) < \infty.$$

If  $\Delta \subset (a-b, a+b)$  then the sequence  $\{|p_n(d\alpha, x)|\}$  is uniformly bounded for  $x \in \Delta$ .

Proof. Let, for simplicity,  $\alpha \in M(0, 1)$ ,  $\Delta = [-\epsilon, \epsilon]$ ,  $0 < \epsilon < 1$ ,  $C_j^{0,1}(d\alpha) = c_j$ . Let us fix  $k = k(\epsilon)$  such that

$$\frac{2}{\sqrt{1-\epsilon}} \sum_{j=k-1}^{\infty} c_j < \frac{1}{2}.$$

If  $m > k$  then by Corollary 5

$$|p_m(x)| \leq \left[ \frac{1}{\sqrt{1-\epsilon}} + 2 \right] |p_k(x)| + |p_{k-1}(x)| + \frac{2}{\sqrt{1-\epsilon}} \sum_{j=k+1}^m c_j |p_j(x)|.$$

Hence

$$|p_m(x)| \leq \left[ \frac{1}{\sqrt{1-\epsilon}} + \frac{1}{2} \right] |p_k(x)| + |p_{k-1}(x)| + \frac{1}{2} \max_{k+1 \leq j \leq m} |p_j(x)|.$$

Thus for  $n > k$

$$\begin{aligned} \max_{k+1 \leq m \leq n} |p_m(x)| &\leq \left[ \frac{1}{\sqrt{1-\epsilon}} + \frac{1}{2} \right] |p_k(x)| + |p_{k-1}(x)| + \frac{1}{2} \max_{k+1 \leq m \leq n} |p_m(x)| \\ &+ \frac{1}{2} \max_{k+1 \leq m \leq n} \max_{k+1 \leq j \leq m} |p_j(x)| = \end{aligned}$$

$$= \left[ \frac{1}{\sqrt{1-\epsilon}} + \frac{1}{2} \right] |p_k(x)| + |p_{k-1}(x)| + \frac{1}{2} \max_{k+1 \leq m \leq n} |p_m(x)|.$$

In particular, for  $n > k$

$$|p_n(x)| \leq \left[ \frac{2}{\sqrt{1-\epsilon}} + 1 \right] |p_k(x)| + |p_{k-1}(x)|.$$

Now remember that  $k$  does not depend on  $n$ .

Sometimes instead of Theorem 3 we shall use the following generalization of the recurrence formula.

Theorem 13. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ ,  $0 \leq n \leq k$ . Then

$$p_n(d\alpha, x) = U_{k-n} \left( \frac{x-a}{b} \right) p_k(d\alpha, x) - U_{k-n-1} \left( \frac{x-a}{b} \right) p_{k+1}(d\alpha, x) + \bar{R}_{n,k}^{a,b}(d\alpha, x)$$

where

$$\bar{R}_{n,k}^{a,b}(d\alpha, x) = \sum_{j=n}^{k-1} U_{j-n} \left( \frac{x-a}{b} \right) \cdot$$

$$\cdot \left\{ \left( 1 - \frac{2}{b} \frac{y_j(d\alpha)}{y_{j+1}(d\alpha)} \right) p_j(d\alpha, x) + \frac{2}{b} [a - \alpha_{j+1}(d\alpha)] p_{j+1}(d\alpha, x) + \left( 1 - \frac{2}{b} \frac{y_{j+1}(d\alpha)}{y_{j+2}(d\alpha)} \right) p_{j+2}(d\alpha, x) \right\}.$$

Proof. The theorem can be proved by induction in exactly the same way as Theorem 3.

Corollary 14. Let  $k \geq 0$ . Then

$$U_{k-1}(x) p_{k+1}(d\alpha, x) = U_k(x) p_k(x) + \sum_{j=0}^{k-1} U_j(x) \left\{ \left( 1 - \frac{2}{b} \frac{y_j(d\alpha)}{y_{j+1}(d\alpha)} \right) p_j(d\alpha, x) - 2\alpha_{j+1}(d\alpha) p_{j+1}(d\alpha, x) + \left( 1 - \frac{2}{b} \frac{y_{j+1}(d\alpha)}{y_{j+2}(d\alpha)} \right) p_{j+2}(d\alpha, x) \right\} - \gamma_0(d\alpha).$$

Theorem 15. Let  $\theta = \theta_1 + i\theta_2$  with  $\theta_2 \leq 0$ . Then  $p_n(d\alpha, x)$  ( $x = \cos \theta$ ) can be represented as

$$(7) \quad \sin \theta p_n(d\alpha, \cos \theta) = |\phi_{2n}(d\alpha, e^{i\theta})| \cdot$$

$$\cdot \sin[(n+1)\theta - \arg \phi_{2n}(d\alpha, e^{i\theta})]$$

or

$$(8) \quad 2i \sin \theta p_n(d\alpha, \cos \theta) =$$

$$= e^{i(n+1)\theta} \phi_{2n}(d\alpha, e^{-i\theta}) - e^{-i(n+1)\theta} \phi_{2n}(d\alpha, e^{i\theta})$$

for  $n = 0, 1, \dots$  where

$$\phi_{2n}(d\alpha, e^{i\theta}) = \sum_{j=0}^n a_j(d\alpha, \cos \theta) e^{ij\theta},$$

$$a_j(d\alpha, x) = \left[ 1 - 2 \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)} \right] p_j(d\alpha, x) - 2\alpha_{j-1}(d\alpha) p_{j-1}(d\alpha, x) + \left[ 1 - \frac{y_{j-2}(d\alpha)}{y_{j-1}(d\alpha)} \right] p_{j-2}(d\alpha, x)$$

( $j = 0, 1, \dots$ ). Consequently  $\phi_{2n}(d\alpha)$  is a polynomial of degree at most  $2n$  with  $\phi_{2n}(d\alpha, 0) = 2^{-n} y_n(d\alpha)$  and  $\phi_{2n}(d\alpha, z^{-1}) = \overline{\phi_{2n}(d\alpha, z)}$  if  $|z| = 1$ .

Proof. Let us write (7) in the form

$$p_n(d\alpha, \cos \theta) = \operatorname{Re} \phi_{2n}(d\alpha, e^{i\theta}) U_n(x) - \operatorname{Im} \phi_{2n}(d\alpha, e^{i\theta}) T_{n+1}(x) (1-x^2)^{-\frac{1}{2}}.$$



### 3.2. Modified Quadrature Processes

Thus (7) means that

$$p_n(d\alpha, x) = R_{n-1}(d\alpha, x) = U_n(x) p_0(d\alpha, x) + R_{n,0}(d\alpha, x)$$

which is equivalent to (2) applied with  $k = 0$ . (8) obviously follows from (7).

Corollary 16. Let  $k$  be a nonnegative integer and let

$$\frac{Y_{j-2}(d\alpha)}{Y_{j-1}(d\alpha)} = \frac{1}{2}, \quad \alpha_{j-1}(d\alpha) = 0$$

for  $j > k$ . Then for each  $n > k$

$$\sin \theta p_n(d\alpha, \cos \theta) = |\phi_{2k}^{i\theta}(d\alpha, e^{i\theta})| \sin[(n+1)\theta - \arg \phi_{2k}^{i\theta}(d\alpha, e^{i\theta})],$$

that is

$$p_n(d\alpha, x) = U_{n-k}(x) p_k(d\alpha, x) - U_{n-k-1}(x) p_{k-1}(d\alpha, x).$$

Let us note that the conditions of corollary 16 are satisfied if  $\text{supp}(d\alpha) = [-1, 1]$  and

$$\alpha(x) = \int_{-1}^x \frac{\sqrt{1-t^2}}{\pi(t)} dt \quad (-1 \leq x \leq 1)$$

where  $\pi$  is a polynomial which is positive on  $[-1, 1]$ . In this case

$$\frac{2}{\pi} |\phi_{2k}^{i\theta}(d\pi, e^{i\theta})|^2 = \pi(\cos \theta)$$

( $\theta \in \mathbb{R}$ ). (See Szegő, Chapter II.)

Lemma 1. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Let  $m$  be a nonnegative integer. If  $n > m-1$  then  $x^m p_{n-1}(d\alpha, x)$  may be written in the form

$$(1) \quad x^m p_{n-1}(d\alpha, x) = R_{m-1,n}^{d\alpha}(x) p_n(d\alpha, x) + a_{n-1,m}^{d\alpha} p_{n-1}(d\alpha, x) + \pi_{n-2,m}^{d\alpha}(x)$$

where  $R_{m-1,n}^{d\alpha}$  and  $\pi_{n-2,m}^{d\alpha}$  are polynomials of degree  $m-1$  and  $n-2$  respectively. Further

$$(2) \quad \lim_{n \rightarrow \infty} a_{n-1,m}^{d\alpha} = \frac{2}{\pi b^2} \int_{a-b}^{a+b} t^m \sqrt{b^2 - (t-a)^2} dt.$$

Proof. Let, for simplicity,  $\alpha \in M(0, 1)$ . For  $m = 0$  the lemma is certainly true. Suppose that for  $m > 1$  we have

$$x^{m-1} p_{n-1}(d\alpha, x) = R_{m-2,n}^{d\alpha}(x) p_n(d\alpha, x) + \sum_{k=n-m}^{n-1} a_{k,m-1}^{d\alpha,n} p_k(d\alpha, x)$$

with existing

$$\lim_{n \rightarrow \infty} a_{k,m-1}^{d\alpha,n} \quad (k = n-m, n-m+1, \dots, n-1)$$

which depends only on  $M(0, 1)$  and is independent of the particular  $\alpha \in M(0, 1)$ . Using the recursion formula we see that

$$x^m p_{n-1}(d\alpha, x) = [x R_{m-2,n}^{d\alpha}(x) + a_{n-1,m-1}^{d\alpha,n} \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)}] p_n(d\alpha, x) + [a_{n-2,m-1}^{d\alpha,n} \frac{Y_{n-2}(d\alpha)}{Y_{n-1}(d\alpha)} + a_{n-1,m-1}^{d\alpha,n} \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)}] p_{n-1}(d\alpha, x) +$$

$$\begin{aligned}
& \sum_{k=n-m+1}^{n-1} \left[ \frac{d\alpha, n}{\alpha_{k-1, m-1}} \frac{Y_{k-1, l}(d\alpha)}{Y_k(d\alpha)} + \frac{d\alpha, n}{\alpha_{k, m-1}} \frac{Y_k(d\alpha)}{Y_{k+1, m-1}} \right] p_k(d\alpha, x) + \\
& + \left[ \frac{d\alpha, n}{\alpha_{n-m, m-1}} \frac{Y_{n-m-1}(d\alpha)}{Y_{n-m}(d\alpha)} + \frac{d\alpha, n}{\alpha_{n-m+1, m-1}} \frac{Y_{n-m}(d\alpha)}{Y_{n-m+1}(d\alpha)} \right] p_{n-m}(d\alpha, x) + \\
& + \frac{d\alpha, n}{\alpha_{n-m, m-1}} \frac{Y_{n-m-1}(d\alpha)}{Y_{n-m}(d\alpha)} p_{n-m-1}(d\alpha, x).
\end{aligned}$$

This formula proves (1) and shows that  $\lim_{n \rightarrow \infty} \alpha_{n-1, m}^{d\alpha}$  exists and depends only on  $M(0, 1)$ . To compute (2) we put in (1)  $\alpha =$  Tschebyshev weight.

We have in this case  $\frac{\pi}{2} p_{n-1}^2(v, x_{kn}(v)) = 1 - x_{kn}^2(v)$ . Thus by the

Gauss-Jacobi mechanical quadrature formula and by (1) we have for

$$m + 3 < 2n$$

$$\begin{aligned}
\frac{2}{\pi} \int_{-1}^1 t^m \sqrt{1-t^2} dt &= \frac{2}{\pi} \sum_{k=1}^n \lambda_{kn}(v) [1 - x_{kn}^2(v)] x_{kn}^m(v) = \\
&= \sum_{k=1}^n \lambda_{kn}(v) p_{n-1}^2(v, x_{kn}(v)) x_{kn}^m(v) = \\
&= \sum_{k=1}^n \lambda_{kn}(v) p_{n-1}^2(v, x_{kn}(v)) + \sum_{k=1}^n \lambda_{kn}(v) p_{n-1}^2(v, x_{kn}(v)) \cdot \\
&\quad \cdot \pi_{n-2, m}^v(x_{kn}(v)) = \alpha_{n-1, m}^v.
\end{aligned}$$

**Lemma 2.** Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then  $[a-b, a+b] \subset \Delta(d\alpha)$ .

**Proof.** It follows from Lemma 1 that if  $f$  is continuous on  $\mathbb{R}$  and has compact support then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) p_{n-1}^2(d\alpha, x_{kn}) = \frac{2}{\pi b} \int_{a-b}^{a+b} f(t) \sqrt{b^2 - (t-a)^2} dt.$$

If  $[a-b, a+b] \not\subset \Delta(d\alpha)$ , then we can choose  $f$  so that  $f(x_{kn}) = 0$  for every  $n$  and  $k = 1, 2, \dots, n$  and

$$\int_{a-b}^{a+b} f(t) \sqrt{b^2 - (t-a)^2} dt < 0$$

which contradicts to the above limit relation.

**Theorem 3.** Let  $\alpha \in M(a, b)$  with  $b > 0$ . Let  $f$  be a complex valued, bounded function on  $\Delta(d\alpha)$ . If  $f$  is Riemann integrable on  $[a-b, a+b]$  then

$$(3) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) p_{n-1}^2(d\alpha, x_{kn}) = \frac{2}{\pi b^2} \int_{a-b}^{a+b} f(t) \sqrt{b^2 - (t-a)^2} dt.$$

**Proof.** If  $f$  is a polynomial then the theorem follows immediately from

Lemma 1. Otherwise we write

$$f = \operatorname{Re}(f) + i \operatorname{Im}(f), \quad \Delta(d\alpha) \setminus \Delta = \Delta + i \operatorname{Im}(f) \setminus \Delta + i \operatorname{Im}(f) \setminus \Delta(d\alpha) \setminus \Delta$$

where  $\Delta = [a-b, a+b]$ . Let, for simplicity,  $\Delta_n(q) = \sum_{k=1}^n \lambda_{kn} q(x_{kn}) p_{n-1}^2(d\alpha, x_{kn})$ . Fix  $\epsilon > 0$ . We construct two polynomials  $\pi_1$  and  $\pi_2$  such that

$$\pi_1(x) \leq \operatorname{Re}(f(x)) \setminus \Delta(x) \leq \pi_2(x)$$

for  $x \in \Delta(d\alpha)$  and

$$\frac{2}{\pi b^2} \int_{\Delta(d\alpha)} [\pi_2(t) - \pi_1(t)] dt < \epsilon.$$

We can do this because  $\operatorname{Re}(f) 1_{\Delta}$  is Riemann integrable on  $\Delta(d\sigma)$  (See

e.g. Szegő, I. 5). Hence

$$\lim_{n \rightarrow \infty} \Lambda_n(\operatorname{Re}(f) 1_{\Delta}) = \frac{2}{\pi b^2} \int_{a-b}^{a+b} \operatorname{Re}(f(t)) \sqrt{b^2 - (t-a)^2} dt.$$

We have, further,

$$|\Lambda_n(\operatorname{Re}(f) 1_{\Delta(d\sigma) \setminus \Delta})| \leq \sup_{t \in \Delta(d\sigma)} |f(t)| \Lambda_n(1_{\Delta(d\sigma) \setminus \Delta})$$

and we can find a polynomial  $\pi$  such that

$$1_{\Delta(d\sigma) \setminus \Delta}(x) \leq \pi(x) \quad (x \in \Delta(d\sigma))$$

and

$$\frac{2}{\pi b} \int_{a-b}^{a+b} \pi(t) dt < \epsilon.$$

Thus

$$\lim_{n \rightarrow \infty} \Lambda_n(\operatorname{Re}(f) 1_{\Delta(d\sigma) \setminus \Delta}) = 0.$$

The limit of  $\Lambda_n(\operatorname{Im}(f))$  can be found in the same way.

Theorem 4. Let  $a \in \mathbb{R}$ ,  $b > 0$ . If for every polynomial  $\pi$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\sigma) \pi(x_{kn}) p_{n-1}^2(d\sigma, x_{kn}) = \frac{2}{\pi b^2} \int_{a-b}^{a+b} \pi(t) \sqrt{b^2 - (t-a)^2} dt$$

then  $\sigma \in M(a, b)$ .

Proof. We have by the recursion formula

$$\sum_{k=1}^n \lambda_{kn}(d\sigma) x_{kn} p_{n-1}^2(d\sigma, x_{kn}) = \sigma_{n-1}(i, \sigma)$$

and

$$\sum_{k=1}^n \lambda_{kn}(d\sigma) x_{kn}^2 p_{n-1}^2(d\sigma, x_{kn}) = \sigma_{n-1}^2(d\sigma) + \frac{\sigma_{n-2}^2(d\sigma)}{\sigma_{n-1}^2(d\sigma)}.$$

The theorem follows immediately from the above identities.

### 3.3. The Support of $d\alpha$

$\text{Supp}(d\alpha)$ , that is the set of points of increase of  $\alpha$ , is always closed. Hence  $\text{supp}(d\alpha)$  is compact iff it is bounded.

**Lemma 1.** The following three statements are equivalent. (i)  $\text{supp}(d\alpha)$

is compact. (ii)  $\sup_{n \in \mathbb{N}} |x_{kn}(d\alpha)| < \infty$ . (iii)  $\sup_{k \in \mathbb{N}} c_k^{0,0}(d\alpha) < \infty$ .

$k=1, \dots, n$

**Proof.** Easy computation. Let us prove e.g. (iii)  $\Rightarrow$  (ii). We have the following important identity by the Gauss-Jacobi mechanical quadrature

formula:

$$\begin{aligned} (1) \quad x_{kn}(d\alpha) &= \lambda_{kn}(d\alpha) \int_{-\infty}^{\infty} x K_n^2(d\alpha, x, x_{kn}) d\alpha(x) = \\ &= \lambda_{kn}(d\alpha) \sum_{j=0}^{n-1} \alpha_j(d\alpha) p_j^2(d\alpha, x_{kn}) + \\ &+ \lambda_{kn}(d\alpha) \sum_{j=1}^{n-1} 2 \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)} p_{j-1}(d\alpha, x_{kn}) p_j(d\alpha, x_{kn}). \end{aligned}$$

Thus

$$|x_{kn}(d\alpha)| \leq \max_{0 \leq j \leq n-1} |\alpha_j(d\alpha)| + 2 \max_{1 \leq j \leq n-1} \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)}.$$

**Lemma 2.** If  $\text{supp}(d\alpha)$  is compact,  $x \in \text{supp}(d\alpha)$  and  $\epsilon > 0$  then there exists a number  $N = N(\epsilon, x)$  such that for every  $n \geq N$   $p_n(d\alpha, t)$  has at least one zero in  $[x-\epsilon, x+\epsilon]$ , in particular,

$$\Delta(d\alpha) = \left[ \lim_{n \rightarrow \infty} x_{nn}(d\alpha), \lim_{n \rightarrow \infty} x_{1n}(d\alpha) \right].$$

Further, if  $\alpha$  is constant on an interval  $\Delta$ , then for every  $n$   $p_n(d\alpha, t)$  has no more than one zero in  $\Delta$ .

**Proof.** See Szegő and Freud.

Note, that if  $\alpha(x) + \alpha(-x) = \text{const}$ ,  $\alpha(x) = \text{const}$  on  $(-1, 1)$  then for every  $n$   $p_{2n+1}(d\alpha, 0) = 0$  but  $p_{2n}(d\alpha, t)$  has no zero in  $(-1, 1)$ .

**Lemma 3.** Let  $\text{supp}(d\alpha)$  be compact. Then

$$\Delta(d\alpha) \subset \left[ \inf_{j \geq 0} \alpha_j - 2 \sup_{j \geq 0} \frac{y_j}{y_{j+1}}, \sup_{j \geq 0} \alpha_j + 2 \sup_{j \geq 0} \frac{y_j}{y_{j+1}} \right]$$

where  $\alpha_j = \alpha_j(d\alpha)$  and  $y_j = y_j(d\alpha)$ .

**Proof.** Let  $A = \inf_{j \geq 0} \alpha_j$ ,  $B = \sup_{j \geq 0} \alpha_j$ . Then by (1)

$$\begin{aligned} x_{kn} - \frac{A+B}{2} &= \lambda_{kn} \sum_{j=0}^{n-1} \left[ \alpha_j - \frac{A+B}{2} \right] p_j^2(d\alpha, x_{kn}) + \\ &+ 2 \lambda_{kn} \sum_{j=1}^{n-1} \frac{y_{j-1}}{y_j} p_{j-1}(d\alpha, x_{kn}) p_j(d\alpha, x_{kn}). \end{aligned}$$

Hence

$$(2) \quad \left| x_{kn} - \frac{A+B}{2} \right| \leq \frac{B-A}{2} + 2 \sup_{j \geq 0} \frac{y_j}{y_{j+1}}.$$

Put here  $k=1$  and let  $n \rightarrow \infty$ . By Lemma 2 we obtain

$$\Delta(d\alpha) \subset \left( -\infty, B + 2 \sup_{j \geq 0} \frac{y_j}{y_{j+1}} \right].$$

If we put  $k=n$  in (2) and let  $n \rightarrow \infty$  then we get



$$\Delta(d\alpha) \subset [A - 2 \sup_{j \geq 0} \frac{Y_j}{Y_{j+1}}, \infty).$$

Lemma 4. Let  $\text{supp}(d\alpha)$  be compact and let  $x$  be fixed. If for every  $\varepsilon > 0$   $\alpha$  takes infinitely many values in  $(x - \varepsilon, x + \varepsilon)$  then there exists a sequence of natural integers  $\{k_n\}_{n=1}^\infty$  such that  $l \leq k_n \leq n$  and

$$(3) \quad \lim_{n \rightarrow \infty} x_{k_n}^{(d\alpha)} = x, \quad \lim_{n \rightarrow \infty} \lambda_{k_n, n}^{(d\alpha)} = 0.$$

Proof. Suppose, without loss of generality, that for every  $\varepsilon > 0$   $\alpha$  takes infinitely many values in  $(x - \varepsilon, x)$ . Let for every  $n$   $j_n$  be defined by  $j_n = \{k: x_{k_n}^{(d\alpha)} < x \leq x_{k_n - l, n}^{(d\alpha)}\}$  with  $x_{0n} = +\infty$ . Let  $k_n = j_n + 1$ . We shall show that  $\{k_n\}_{n=1}^\infty$  satisfies the requirements of the lemma. Because of Lemma 2  $k_n < n$  for  $n$  large. If we can show that

$$(4) \quad \lim_{n \rightarrow \infty} x_{k_n + l, n} = x$$

then

$$\lim_{n \rightarrow \infty} x_{k_n}^{(d\alpha)} = \lim_{n \rightarrow \infty} x_{j_n}^{(d\alpha)} = x$$

and by the Markov-Stieltjes inequalities

$$\lambda_{k_n, n}^{(d\alpha)} \leq \int_{x_{k_n + l, n}}^{x_{k_n - l, n}} d\alpha(t) \xrightarrow{n \rightarrow \infty} \alpha(x - 0) - \alpha(x + 0) = 0.$$

Suppose now that (4) does not hold. Then there exists an  $\varepsilon > 0$  and a sequence  $\{n_l\}$  such that  $p_{n_l}^{(d\alpha, t)}$  has no more than two zeros in  $(x - \varepsilon, x)$  for  $l = 1, 2, \dots$ . Because  $\alpha$  takes infinitely many values in

$(x - \varepsilon, x)$  we can find three points  $x_1, x_2, x_3 \in (x - \varepsilon, x) \cap \text{supp}(d\alpha)$  and by Lemma 2  $p_{n_l}^{(d\alpha, t)}$  must have zeros near each  $x_k$  for every  $l$  large. Hence  $p_{n_l}^{(d\alpha, t)}$  has at least three zeros in  $(x - \varepsilon, x)$ . This contradiction proves (4).

Lemma 5. Let  $\text{supp}(d\alpha)$  be compact and let  $x \in \mathbb{R}$  be fixed. Suppose that for every  $\varepsilon > 0$   $\alpha$  takes infinitely many values in  $(x - \varepsilon, x + \varepsilon)$ . Then for every  $c \in \mathbb{R}$

$$|x - c| \leq \limsup_{j \rightarrow \infty} |\alpha_j(d\alpha) - c| + 2 \limsup_{j \rightarrow \infty} \frac{Y_{j-1}(d\alpha)}{Y_j(d\alpha)},$$

in particular, if  $a = \lim_{j \rightarrow \infty} \alpha_j(d\alpha)$  exists, then

$$x \in \left\{ a - 2 \limsup_{j \rightarrow \infty} \frac{Y_{j-1}}{Y_j}, a + 2 \limsup_{j \rightarrow \infty} \frac{Y_{j-1}}{Y_j} \right\}.$$

Proof. Let us take  $\{k_n\}$  from Lemma 4. Let  $M \in \mathbb{N}$ . Then by (1)

$$\begin{aligned} |x_{k_n, n} - c| &\leq \lambda_{k_n, n} \sum_{j=0}^{M-1} |\alpha_j - c| p_j^2(d\alpha, x_{k_n, n}) + \\ &+ \sup_{j \geq M} |\alpha_j - c| + 2 \lambda_{k_n, n} \sum_{j=1}^{M-1} \frac{Y_{j-1}}{Y_j} |p_{j-1}(d\alpha, x_{k_n, n}) p_j(d\alpha, x_{k_n, n})| + \\ &+ 2 \sup_{j \geq M} \frac{Y_{j-1}}{Y_j}. \end{aligned}$$

First let  $n \rightarrow \infty$ , then  $M \rightarrow \infty$ .

Lemma 6. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then  $[a - b, a + b] \subset \text{supp}(d\alpha)$ .

Proof. If  $[a-b, a+b] \not\subset \text{supp}(d\alpha)$  then  $[a-b, a+b] \cap [\mathbb{R} \setminus [a-b, a+b] \cap \text{supp}(d\alpha)]$  contains an interval  $\Delta_1$ . Let  $\Delta \subset \Delta_1^0$ . Then by Theorem 3.2.3

$$(5) \quad \sum_{x_{kn} \in \Delta} \lambda_{kn}^2 p_{n-1}^2(d\alpha, x_{kn}) \xrightarrow{n \rightarrow \infty} \frac{2}{n^2} \int_{\Delta} \frac{2}{\sqrt{b^2 - (t-a)^2}} dt > 0.$$

On the other hand by Lemma 2  $\Delta$  contains no more than one  $x_{kn}$  for every  $n$  since  $\Delta \cap \text{supp}(d\alpha) = \emptyset$ . Further  $\Delta \subset (a-b, a+b)$  and by Theorem 3.1.9

$$\lim_{n \rightarrow \infty} \lambda_{kn}(d\alpha, x) p_{n-1}^2(d\alpha, x) = 0$$

uniformly for  $x \in \Delta$ . Thus the left side of (5) converges to 0 when  $n \rightarrow \infty$ . This contradiction shows that  $[a-b, a+b] \subset \text{supp}(d\alpha)$ .

Theorem 7. Let  $\text{supp}(d\alpha)$  be compact and let  $\lim_{j \rightarrow \infty} \alpha_j(d\alpha) = a$  exist. Then

$$\text{supp}(d\alpha) = A \cup B, \quad A \cap B = \emptyset$$

where  $A$  is closed and belongs to

$$(6) \quad \left[ a - 2 \limsup_{j \rightarrow \infty} \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)}, a + 2 \limsup_{j \rightarrow \infty} \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)} \right],$$

$B$  is denumerable, isolated, the only two possible limit points of  $B$  are the two endpoints of (6), if  $x \in B$  then  $\alpha_{ac} + \alpha_s$  is constant near  $x$  and  $\alpha$  has an isolated jump at  $x$ , furthermore,

$$B \subset \left\{ \inf_{j \geq 0} \alpha_j - 2 \sup_{j \geq 1} \frac{y_{j-1}}{y_j}, \sup_{j \geq 0} \alpha_j + 2 \sup_{j \geq 1} \frac{y_{j-1}}{y_j} \right\}.$$

If  $\alpha \in M(a, b)$  then  $A$  is the interval (6).

Proof. The theorem follows immediately from Lemmas 1-6. The only thing which we have to show is that if  $\alpha \in M(a, 0)$  then  $a \in \text{supp}(d\alpha)$ . If

$a \notin \text{supp}(d\alpha)$  then  $B = \text{supp}(d\alpha)$  and hence  $B$  is closed. But  $B$  can be closed only if  $B$  is finite and then  $\alpha$  has only finitely many points of increase, that is  $\alpha$  is not a weight.

Theorem 8. Let  $\alpha \in M(a, b)$ . If  $x \notin \text{supp}(d\alpha)$  then there exist  $\varepsilon > 0$  and  $N \geq 0$  such that for every  $n \geq N$   $p_n(d\alpha, t)$  has no zeros in  $[x-\varepsilon, x+\varepsilon]$ .

Proof. Let, for simplicity,  $a = 0$ . By Theorem 7  $x \notin \text{supp}(d\alpha)$  implies  $x \notin [-b, b]$  (or  $x \neq 0$  if  $b = 0$ ). Suppose, without loss of generality, that  $b < x < \infty$ . If  $x \notin \Delta(d\alpha)$  then the theorem says nothing since  $x_{kn}(d\alpha) \in \Delta(d\alpha)$  for every  $n$  and  $1 \leq k \leq n$ . Now let  $x \in \Delta(d\alpha) \cap (b, \infty)$ . By Theorem 7  $(\frac{b+x}{2}, \infty) \cap \text{supp}(d\alpha)$  is finite and it is not empty since  $x \in \Delta(d\alpha)$ . Let  $t_1 < t_2 < \dots < t_m$  denote those points of  $\text{supp}(d\alpha)$  which belong to  $(x, \infty)$ . Let  $\varepsilon > 0$  be such that

$$x + \varepsilon < t_1 - \varepsilon < t_1 + \varepsilon < t_2 - \varepsilon < \dots < t_{m-1} + \varepsilon < t_m - \varepsilon < t_m + \varepsilon$$

and  $[x-\varepsilon, x] \cap \text{supp}(d\alpha) = \emptyset$ . By Lemma 2 we can find  $N = N(\varepsilon, \{t_i\})$  such that for every  $n \geq N(\varepsilon)$   $p_n(d\alpha, t)$  has zeros in each  $[t_i - \varepsilon, t_i + \varepsilon]$  ( $i = 1, 2, \dots, m$ ). Thus for  $n \geq N(\varepsilon)$   $p_n(d\alpha, t)$  has not less than  $m$  zeros in  $[t_1 - \varepsilon, \infty)$ . On the other hand  $\alpha$  takes exactly  $m+1$  values in  $(x-\varepsilon, x)$  if we do not count the values of  $\alpha(t_i)$ . Further,  $[t_m, \infty)$  does not contain zeros of  $p_n(d\alpha, t)$  since  $\Delta(d\alpha) \cap (t_m, \infty) = \emptyset$ . Thus by Lemma 2 for every  $n \geq 0$   $p_n(d\alpha, t)$  has no more than  $m$  zeros in  $(x-\varepsilon, \infty)$ . Hence for  $n \geq N$  both  $(x-\varepsilon, \infty)$  and  $[t_1 - \varepsilon, \infty)$  contain exactly  $m$  zeros of  $p_n(d\alpha, t)$ , that is  $[x-\varepsilon, x+\varepsilon]$  contains no zeros of  $p_n(d\alpha, t)$  if  $n \geq N$ .



Let us note that without the assumption  $\alpha \in M(a, b)$  Theorem 8 does not necessarily hold. (See Remark 4.1.6.)

#### 4. Limit Relations

##### 4.1. Pointwise Limits

We begin with a simple result which we shall not apply in the following but which is worth recording.

Theorem 1. For every weight  $\alpha$  and  $x \in \mathbb{R}$

$$(1) \quad \sum_{k=0}^{\infty} \lambda_{k+1}^2(d\alpha, x) p_k^2(d\alpha, x) \leq [1 + \alpha(x) - \alpha(-\infty)]^2,$$

in particular, for every  $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \lambda_{n+1}(d\alpha, x) p_n(d\alpha, x) = 0.$$

Proof. Let  $x$  be fixed and let  $\beta = \alpha + \delta_x$  where  $\delta_x$  is the unit mass concentrated at  $x$ . Let us expand  $p_n(d\alpha, t)$  in a Fourier series in  $p_k(d\beta, t)$ . We have

$$p_n(d\alpha, t) = \frac{Y_n(d\beta)}{Y_n(d\alpha)} p_n(d\beta, t) + K_{n+1}(d\beta, t, x) p_n(d\alpha, x).$$

Putting  $t = x$  we obtain

$$p_n(d\alpha, x) = \frac{Y_n(d\beta)}{Y_n(d\alpha)} p_n(d\beta, x) + \lambda_{n+1}^{-1}(d\alpha, x) p_n(d\alpha, x).$$

By an easy computation  $\lambda_{n+1}(d\beta, x) = \lambda_{n+1}(d\alpha, x) + 1$ ,  $Y_n(d\beta) \leq Y_n(d\alpha)$  and  $\lambda_{n+1}(d\alpha, x) \leq \alpha(x) - \alpha(-\infty)$ . Thus for every  $n = 0, 1, \dots$

$$\lambda_{n+1}^2(d\alpha, x) p_n^2(d\alpha, x) \leq p_n^2(d\beta, x) [1 + \alpha(x) - \alpha(-\infty)]^2$$

and

$$\sum_{k=0}^{n-1} \lambda_{k+1}^2(d\alpha, x) p_k^2(d\alpha, x) \leq \lambda_n^{-1}(d\beta, x) [1 + o(x) - o(-\infty)]^2 \leq \\ \leq [1 + o(x) - o(-\infty)]^2.$$

Letting  $n \rightarrow \infty$  we obtain (1).

Lemma 2. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then for every  $x \in [a-b, a+b]$

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}^2(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} = +\infty.$$

Proof. Let, for simplicity,  $a \leq x \leq a+b$ . Let  $0 < x \leq a+b$ . Let  $0 < \varepsilon < b$ . Then

$$\sum_{k=1}^n \lambda_{kn}^2(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \geq \varepsilon^{-2} \sum_{x-\varepsilon \leq x_{kn} \leq x} \lambda_{kn}^2 p_{n-1}^2(d\alpha, x_{kn})$$

and by Theorem 3.2.3

$$\liminf_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}^2(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \geq \frac{2}{\varepsilon^2} \int_{x-\varepsilon}^x \sqrt{b-(t-a)}^2 dt.$$

If  $x < a+b$  the right side is exactly of order  $\varepsilon^{-1}$ , if  $x = a+b$  it is exactly of order  $\varepsilon^{-\frac{1}{2}}$ . Letting  $\varepsilon \rightarrow 0$  we obtain (2).

Theorem 3. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then for  $x \in [a-b, a+b]$

$$(3) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) p_n^2(d\alpha, x) = 0$$

and the convergence is uniform for  $x \in \Delta \subset (a-b, a+b)$ .

Proof. To get (3) use Lemma 2 and the formula

$$\lambda_n(d\alpha, x) p_n^2(d\alpha, x) = \frac{p_n^2(d\alpha)}{p_{n-1}^2(d\alpha)} \left[ \sum_{k=1}^n \frac{\lambda_{kn}^2(d\alpha) p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \right]^{-1}$$

which follows from  $\lambda_n^{-1}(d\alpha, x) = \sum_{k=1}^n \frac{p_{kn}^2(d\alpha, x)}{\lambda_{kn}^2(d\alpha)}$ . The uniform convergence inside (a-b, a+b) follows from Theorem 3.1.9 and from

$$p_n^2(d\alpha, x) \leq C [p_{n-1}^2(d\alpha, x) + p_{n-2}^2(d\alpha, x)]$$

for  $x \in \Delta(d\alpha)$  which can easily be proved using the recurrence formula.

There are two possible ways to define the Christoffel functions for complex values of the argument. We can either put

$$\lambda_n(d\alpha, z) = \left[ \sum_{k=0}^{n-1} p_k^2(d\alpha, z) \right]^{-1}$$

or

$$\lambda_n(d\alpha, z) = \left[ \sum_{k=0}^{n-1} |p_k(d\alpha, z)|^2 \right]^{-1}.$$

It is easy to see that the second definition coincides with

$$\lambda_n(d\alpha, z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |1 + (z-t) \pi_{n-2}(t)|^2 d\alpha(t).$$

To avoid confusion we shall write  $\lambda_n^*(d\alpha, z)$  when we mean the first definition:

$$\lambda_n^*(d\alpha, z) = \left[ \sum_{k=0}^{n-1} p_k^2(d\alpha, z) \right]^{-1}.$$

Let for  $z, u \in \mathbb{C}$

$$K_n(d\alpha, z, u) = \sum_{k=0}^{n-1} p_k(d\alpha, z) \overline{p_k(d\alpha, u)},$$

$$k_n(d\alpha, z, u) = \sum_{k=0}^{n-1} p_k(d\alpha, z) p_k(d\alpha, u).$$

Properties 4.  $\lambda_n$  is real valued, monotonic in  $n$  and positive,  $\lambda_n^*$  is monomorphic with  $2n-2$  poles.

$$\lambda_n^{-1}(z) = K_n(z, z), K_n(z, u) = \overline{K_n(u, z)}, \lambda_n^{-1}(t) = k_n(z, z), k_n(z, u) = k_n(u, z),$$

$$K_n(d\alpha, z, u) = \sum_{k=1}^n \frac{f_k(d\alpha, z) \overline{f_k(d\alpha, u)}}{\lambda_{kn}(d\alpha)},$$

$$k_n(d\alpha, z, u) = \sum_{k=1}^n \frac{f_k(d\alpha, z) f_k(d\alpha, u)}{\lambda_{kn}(d\alpha)},$$

further

$$(4) \quad \lambda_n^{-1}(d\alpha, z) = |p_n(d\alpha, z)|^2 \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{|z - x_{kn}|^2},$$

and

$$(5) \quad \lambda_n^*(d\alpha, z)^{-1} = p_n^2(d\alpha, z) \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(z - x_{kn})^2}.$$

We obtain immediately from (4) and (5) the following

Theorem 5. Let  $\text{supp}(d\alpha)$  be compact and let  $z \notin \Delta(d\alpha)$ . Then

$$\liminf_{n \rightarrow \infty} \lambda_n(d\alpha, z) |p_n^2(d\alpha, z)| > 0$$

and

$$\liminf_{n \rightarrow \infty} |\lambda_n^*(d\alpha, z) p_n^2(d\alpha, z)| > 0.$$

Remark 6. It is not true that Theorem 5 holds for every  $z \notin \text{supp}(d\alpha)$ . For, if  $\alpha(x) + \alpha(-x) = \text{const}$ ,  $\text{supp}(d\alpha)$  is compact and  $\alpha(t) = \text{const}$  for  $t \in [-\epsilon, \epsilon]$  ( $\epsilon > 0$ ) then  $p_{2k+1}(d\alpha, 0) = 0$  although  $0 \notin \text{supp}(d\alpha)$ .

Theorem 7. Let  $\alpha \in M(a, 0)$ . Then

(i) for every  $z \notin \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \lambda_n^*(d\alpha, z) p_n^2(d\alpha, z) = \alpha, \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, z) p_n^2(d\alpha, z) = \alpha.$$

(ii) for every  $x \in \text{supp}(d\alpha) \setminus a$

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) p_n^2(d\alpha, x) = 0.$$

(iii) there exist two weights  $\hat{\alpha}$  and  $\hat{a}$  in  $M(a, 0)$  such that

$$\lim_{n \rightarrow \infty} \lambda_n(d\hat{\alpha}, a) p_n^2(d\hat{\alpha}, a) = 0$$

and

$$\liminf_{n \rightarrow \infty} \lambda_n(d\hat{\alpha}, a) p_n^2(d\hat{\alpha}, a) = 0,$$

$$\limsup_{n \rightarrow \infty} \lambda_n(d\hat{\alpha}, a) p_n^2(d\hat{\alpha}, a) = \infty.$$

Proof. (i) If  $z$  is complex use (4) and (5). Let now  $z = x$  be real and  $x \notin \text{supp}(d\alpha)$ . Since  $\text{supp}(d\alpha)$  is compact and  $\alpha$  is constant in a neighborhood of  $x$ , we have  $\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) = 0$ . Suppose that there exists a sequence  $n_1 < n_2 < \dots$  such that

$$\lim_{k \rightarrow \infty} \lambda_n(d\alpha, x) p_n^2(d\alpha, x) < \infty.$$

We have by the recurrence formula

$$\begin{aligned} x \lambda_n^{-1}(x) &= \sum_{k=0}^{n-1} \alpha_k p_k^2(x) + 2 \sum_{k=0}^{n-2} \frac{Y_k}{Y_{k+1}} p_k(x) p_{k+1}(x) + \\ &+ \frac{Y_{n-1}}{Y_n} p_{n-1}(x) p_n(x) \end{aligned}$$

with  $\lambda_n(x) = \lambda_n(d\alpha, x)$ ,  $\alpha_k = \alpha_k(d\alpha)$  etc. Let  $M$  be a natural integer.

Then

$$\begin{aligned} |x-a| &\leq \lambda_n(x) \sum_{k=0}^{M-1} |\alpha_k - a| p_k^2(x) + \sup_{k \geq M} |\alpha_k - a| + \\ &+ 2 \lambda_n(x) \sum_{k=0}^{M-1} \frac{Y_k}{Y_{k+1}} |p_k(x) p_{k+1}(x)| + 2 \sup_{k \geq M} \frac{Y_k}{Y_{k+1}} + \\ &+ \frac{Y_{n-1}}{Y_n} |\lambda_n(x) p_n^2(x)|^{\frac{1}{2}}. \end{aligned}$$

Put  $n = n_f$ , first let  $f \rightarrow \infty$  and then  $M \rightarrow \infty$ . We get  $|x-a| \leq 0$  that is  $x = a$ . By Theorem 3.37  $x \in \text{supp}(d\alpha)$ . This is a contradiction.

(ii) If  $x \in \text{supp}(d\alpha) \setminus a$  then by Theorem 3.3.7  $x$  is a jump of  $\alpha$  and consequently (ii) is true.

(iii) Let  $\hat{\alpha}$  be defined by  $\alpha_i(d\hat{\alpha}) \equiv a$  and  $Y_i(d\hat{\alpha})/Y_{i+1}(d\hat{\alpha}) = (i+1)^{-\frac{1}{2}}$  for  $i = 0, 1, \dots$ . Then  $\hat{\alpha}(a+x) + \hat{\alpha}(a-x) = \text{const}$  and thus  $p_{2k+1}(d\hat{\alpha}, a) = 0$  for  $k = 0, 1, \dots$ . Hence

$$(6) \quad \lim_{k \rightarrow \infty} \lambda_{2k+1}(d\hat{\alpha}, a) p_{2k+1}^2(d\hat{\alpha}, a) = 0.$$

-37-

Let us consider now  $p_{2k}(d\hat{\alpha}, a)$ . By the recurrence formula

$$\frac{Y_{2k-1}}{Y_{2k}} p_{2k}(a) + \frac{Y_{2k-2}}{Y_{2k-1}} p_{2k-2}(a) = 0.$$

Hence

$$(7) \quad p_{2k}^2(a) = \left(\frac{Y_{2k-1}}{Y_{2k-2}}\right)^2 p_{2k-2}^2(a).$$

By repeating application of (7) we obtain that for every  $j = 1, 2, \dots, k$

$$p_{2k}^2(a) = \left[\frac{Y_{2k-1}}{Y_{2k-2}} \frac{Y_{2k-2}}{Y_{2k-3}} \dots \frac{Y_{2k-2(j-1)}}{Y_{2k-2(j-1)-1}}\right]^2 p_{2k-2j}^2(a).$$

Thus for  $j = 0, 1, \dots, k-1$

$$p_{2k}^2(a) \leq \sqrt{2k} p_{2j}^2(a),$$

that is

$$p_{2k}^2(a) \leq \sqrt{\frac{2}{k}} \sum_{j=0}^{k-1} p_{2j}^2(a) = \sqrt{\frac{2}{k}} \lambda_{2k}^{-1}(a)$$

which together with (6) proves the first part of (iii). Let  $\hat{\alpha}$  be defined by  $\alpha_i(d\hat{\alpha}) \equiv a$  and  $Y_i(d\hat{\alpha})/Y_{i+1}(d\hat{\alpha}) = \exp(-(i+1)^2)$  for  $i = 0, 1, \dots$ .

Repeating the above argument we see that (6) holds if we replace  $\hat{\alpha}$  by  $\hat{\alpha}$ . Further, similarly to (7),

$$p_{2k}^2(d\hat{\alpha}, a) = e^{8k-2} p_{2k-2}^2(d\hat{\alpha}, a).$$

Thus

-38-



$$p_{2k}^2(\hat{a}, a) \geq \frac{e^{8k-2}}{k} \lambda_{2k}^{-1}(\hat{a}, a).$$

Let now  $k \rightarrow \infty$ .

Let us remark that both  $\hat{a}$  and  $\hat{\alpha}$  are continuous at  $a$ .

Definition 8. For  $z \in \mathbb{C} \setminus [-1, 1]$  we define  $\rho(z)$  by

$$\rho(z) = z + \sqrt{z^2 - 1}$$

where we take that branch of  $\sqrt{z^2 - 1}$  for which  $|\rho(z)| > 1$  whenever  $z \in \mathbb{C} \setminus [-1, 1]$ . We have

$$\lim_{z \rightarrow \infty} \rho(z) = \infty, \quad \lim_{z \rightarrow \infty} |z/\rho(z)| = \frac{1}{2}.$$

Lemma 9. Let  $|z| > 1$ . If

$$\lim_{k \rightarrow \infty} a_k z^{-k} = a$$

then

$$\lim_{n \rightarrow \infty} \frac{z-1}{z^{n+1}} \sum_{k=0}^n a_k = a.$$

Proof. The matrix  $\mu = \{\mu_{kn}\}$  where  $\mu_{kn} = (z-1)z^{k-n-1}$  satisfies the conditions of Toeplitz-Silverman's theorem.

Lemma 10. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$  and let  $v_\tau$  denote the Tschelyshev weight corresponding to  $\tau = [a-b, a+b]$ . Then for every  $z \in \mathbb{C} \setminus [a-b, a+b]$

$$\lim_{n \rightarrow \infty} \lambda_n(v_\tau, t) |p_n^2(v_\tau, t)| = \left| \rho\left(\frac{z-a}{b}\right) \right|^2 - 1$$

and

$$\lim_{n \rightarrow \infty} \lambda_n^*(v_\tau, z) |p_n^2(v_\tau, z)| = \rho\left(\frac{z-a}{b}\right)^2 - 1.$$

Proof. Use Lemma 9 and the formula

$$p_n(v_\tau, z) = \frac{1}{\sqrt{2\pi}} \left[ \rho\left(\frac{z-a}{b}\right)^n + \rho\left(\frac{z-a}{b}\right)^{-n} \right]$$

for  $n = 1, 2, \dots$

Theorem 11. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then

(i) for every  $z \notin \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, z) |p_n^2(d\alpha, z)| = \left| \rho\left(\frac{z-a}{b}\right) \right|^2 - 1$$

and

$$\lim_{n \rightarrow \infty} \lambda_n^*(d\alpha, z) |p_n^2(d\alpha, z)| = \rho\left(\frac{z-a}{b}\right)^2 - 1,$$

(ii) for every  $x \in \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) |p_n^2(d\alpha, x)| = 0,$$

(iii) the convergence in (ii) is uniform inside  $(a-b, a+b)$ .

Proof. (i) If  $z \notin \Delta(d\alpha)$  then both  $|z-t|^{-2}$  and  $(z-t)^{-2}$  are continuous functions of  $t \in \Delta(d\alpha)$ . Thus by (4), (5) and Theorem 3.2.3

$$(8) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, z) |p_n^2(d\alpha, z)| = \left| \frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2 - (t-a)^2}}{|z-t|^2} dt \right|^{-1}$$

and

$$(9) \quad \lim_{n \rightarrow \infty} \lambda_n^*(d\alpha, z) |p_n^2(d\alpha, z)| = \left| \frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2 - (t-a)^2}}{(z-t)^2} dt \right|^{-1}.$$

If  $z \in \Delta(d\alpha)$  but  $z \notin \text{supp}(d\alpha)$  then we take  $\varepsilon$  from Theorem 3.3.8 and put

$$f_1(t) = \begin{cases} |z-t|^{-2} & \text{for } |z-t| > \varepsilon \\ 0 & \text{for } |z-t| \leq \varepsilon, \end{cases}$$

$$f_2(t) = \begin{cases} (z-t)^{-2} & \text{for } |z-t| > \varepsilon \\ 0 & \text{for } |z-t| \leq \varepsilon. \end{cases}$$

Both  $f_1$  and  $f_2$  satisfy the conditions of Theorem 3.2.3. By Theorem 3.3.8 neither (4) nor (5) will change if we replace  $|z-t|^{-2}$  and  $(z-t)^2$  by  $f_1(t)$  and  $f_2(t)$  respectively for  $n \geq N$ . Thus (8) and (9) hold for every  $z \notin \text{supp}(d\alpha)$ . To calculate the integrals on the right sides of (8) and (9) let us remark that it is the same for every  $\alpha \in M(a, b)$ , in particular, for the Tschebyshev weight corresponding to  $[a-b, a+b]$ . Now we use Lemma 10.

(ii) If  $x \in [a-b, a+b]$  then use Theorem 3. If  $x \in \text{supp}(d\alpha) \setminus [a-b, a+b]$  then by Theorem 3.3.7  $\alpha$  has a jump at  $x$  which implies (ii) again.

(iii) See Theorem 3.

Theorem 12. Let  $\text{supp}(d\alpha)$  be compact and let  $\alpha \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ . If there exists a sequence  $\{z_k\}_{k=1}^\infty$  such that  $z_k \in \mathbb{C}$ ,  $\lim_{k \rightarrow \infty} z_k = \alpha$  and

$$\lim_{k \rightarrow \infty} \frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)} = \rho\left(\frac{z-\alpha}{b}\right)^{-1}$$

for  $k = 1, 2, \dots$  then  $\alpha \in M(a, b)$ .

Proof. Suppose without loss of generality that  $z_k \notin \Delta(d\alpha)$  for every  $k$ . We have

$$(10) \quad \frac{z p_{n-1}^{(a, z)}}{p_n(d\alpha, z)} = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \left[ 1 + \sum_{k=1}^n \lambda_{kn}(d\alpha) x_{kn} \frac{p_{n-1}^{(d\alpha, x_{kn})}}{z - x_{kn}} \right]$$

which can easily be checked. Let  $d(z) = \text{dist}(z, \Delta(d\alpha))$ . Then we get with  $C = C(\text{supp}(d\alpha))$

$$\left| \frac{z p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)} \right| \leq \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} [1 + C d(z_k)^{-1}].$$

Letting first  $n \rightarrow \infty$  and then  $k \rightarrow \infty$  we obtain

$$\frac{b}{2} \leq \liminf_{n \rightarrow \infty} \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)}.$$

On the other hand we have by the recurrence formula and Lemma 3.3.1

$$\frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \leq \left[ |z_k| + C_1 \right] \frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)} + C_2 \left| \frac{p_{n-2}(d\alpha, z_k)}{p_n(d\alpha, z_k)} \right|$$

where  $C_1$  and  $C_2$  depend on  $\text{supp}(d\alpha)$ . First let  $n \rightarrow \infty$  and then  $k \rightarrow \infty$ . We get

$$\limsup_{n \rightarrow \infty} \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \leq \frac{b}{2}.$$

Using again the recurrence formula we obtain

$$\alpha_n(d\alpha) = z_1 \cdot \frac{Y_n(d\alpha)}{Y_{n+1}(d\alpha)} \frac{p_{n+1}(d\alpha, z_1)}{p_n(d\alpha, z_1)} - \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \frac{p_{n-1}(d\alpha, z_1)}{p_n(d\alpha, z_1)}.$$

Thus  $\alpha_n(d\alpha)$  is convergent and letting  $n \rightarrow \infty$  we see that

$$\lim_{n \rightarrow \infty} \alpha_n(d\alpha) = z_1 - \frac{b}{2} \rho\left(\frac{z_1 - \alpha}{b}\right)^{-1} - \frac{b}{2} \rho\left(\frac{z_1 - \alpha}{b}\right)^{-1} = \alpha.$$

Theorem 13. Let  $\alpha \in M(a, b)$  and let  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ . Then



$$\lim_{n \rightarrow \infty} \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} = \begin{cases} 0 & \text{for } b = 0 \\ \rho(\frac{z-a}{b})^{-1} & \text{for } b > 0 \end{cases}$$

Proof. If  $b = 0$  then the theorem follows immediately from (10) and

Theorem 3.3.8. If  $b > 0$  then by (10) and Theorems 3.2.3 and 3.3.8

$$\lim_{n \rightarrow \infty} \frac{zp_{n-1}(d\alpha, z)}{p_n(d\alpha, z)}$$

exists for  $z \notin \text{supp}(d\alpha)$  and equals to

$$\frac{z}{\pi b} \int_{a-b}^{a+b} \frac{\sqrt{b^2 - (t-a)^2}}{z-t} dt.$$

It can directly be calculated that the latter expression equals  $z\rho(\frac{z-a}{b})^{-1}$  but it is easier if we remark that  $zp_{n-1}(v_r, z)/p_n(v_r, z)$  converges to the same limit when  $v_r$  is the Tschebyshev weight corresponding to  $\tau = [a-b, a+b]$ .

Using Theorems 7, 11 and 13 we can prove limit relations for

$\lambda_{n+1}(d\alpha, z)/p_n^2(d\alpha, z)$  and  $\lambda_{n+1}^*(d\alpha, z)/p_n^2(d\alpha, z)$  as well. We will not go into details, we only formulate one result which we shall use later.

Theorem 14. Let  $\alpha \in M(a, b)$  with  $b > 0$  and let  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ . Then

$$\lim_{n \rightarrow \infty} \lambda_{n+1}^*(d\alpha, z)/p_n^2(d\alpha, z) = 1 - \rho(\frac{z-a}{b})^{-2}.$$

Lemma 15. Let  $\alpha \in M(a, b)$  and  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ . Then

$$\lim_{n \rightarrow \infty} \frac{p'_n(d\alpha, z)}{p_n(d\alpha, z)} = \begin{cases} \frac{1}{z-a} & \text{for } b = 0 \\ \sqrt{\frac{1}{(z-a)^2 - b^2}} & \text{for } b > 0 \end{cases}$$

where  $\sqrt{z^2 - 1} > 0$  for  $z > 1$ .

Proof. We have for  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\frac{p'_n(d\alpha, z)}{p_n(d\alpha, z)} = \sum_{k=1}^n \frac{1}{z - x_{kn}(d\alpha)}.$$

Let  $\epsilon > 0$  and

$$f(t) = \begin{cases} 0 & \text{for } |z-t| < \epsilon \\ \frac{1}{z-t} & \text{for } |z-t| \geq \epsilon \end{cases}$$

By Theorem 3.3.8 if  $\epsilon = \epsilon(z)$  is small enough and  $n \geq N(z)$  then

$$\frac{p'_n(d\alpha, z)}{p_n(d\alpha, z)} = \frac{1}{n} \sum_{k=1}^n f(x_{kn}(d\alpha)).$$

Using Theorems 5.2 and 5.3 we obtain

$$\lim_{n \rightarrow \infty} \frac{p'_n(d\alpha, z)}{p_n(d\alpha, z)} = \begin{cases} f(a) & \text{for } b = 0 \\ \frac{1}{\pi} \int_{a-b}^{a+b} \frac{f(t)}{\sqrt{b^2 - (t-a)^2}} dt & \text{for } b > 0 \end{cases}$$

and  $f(a) = (z-a)^{-1}$  for  $b = 0$  and  $f(t) = (z-t)^{-1}$  for  $b > 0$ ,  $t \in [a-b, a+b]$ .

The calculation of the above integral is simple; put  $\alpha = \text{Tschebyshev}$

weight corresponding to  $[a-b, a+b]$ .

Theorem 16. Let  $\alpha \in M(a, b)$  and  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ . Then

$$\lim_{n \rightarrow \infty} n \left[ \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} - \frac{p'_n(d\alpha, z)}{p'_n(d\alpha, z)} \right] = \begin{cases} 0 & \text{for } b = 0 \\ \rho \left( \frac{z-a}{b} \right)^{-1} & \text{for } b > 0. \end{cases}$$

Proof. From

$$p_{n-1}(d\alpha, z) = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} p_n(d\alpha, z) \sum_{k=1}^n \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{z - x_{kn}}$$

follows

$$\begin{aligned} n \left[ \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} - \frac{p'_n(d\alpha, z)}{p'_n(d\alpha, z)} \right] &= \\ &= n \frac{p_n(d\alpha, z)}{p'_n(d\alpha, z)} \cdot \frac{Y_{n-1}}{Y_n} \sum_{k=1}^n \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(z - x_{kn})^2}. \end{aligned}$$

If  $b = 0$  then use Theorem 3.3.8 and Lemma 15. If  $b > 0$  then use

Theorems 3.2.3, 3.3.8 and Lemma 15. For  $b > 0$  we get

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left[ \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} - \frac{p'_n(d\alpha, z)}{p'_n(d\alpha, z)} \right] &= \\ &= \sqrt{\left( \frac{z-a}{b} \right)^2 - 1} \cdot \frac{1}{\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2 - (t-a)^2}}{(z-t)^2} dt \end{aligned}$$

and this integral has been calculated in the course of proof of Theorem 11.

From Theorems 13 and 16 we obtain

Theorem 17. Let  $\alpha \in M(a, b)$  and  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ . Then

-45-

$$\lim_{n \rightarrow \infty} \frac{p'_{n-1}(d\alpha, z)}{p'_n(d\alpha, z)} = \begin{cases} 0 & \text{for } b = 0 \\ \rho \left( \frac{z-a}{b} \right)^{-1} & \text{for } b > 0. \end{cases}$$

The following result is rather surprising if we compare it with Theorem 13.

Theorem 18. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Then for every

$x \in \text{supp}(d\alpha) \setminus [a-b, a+b]$

$$\lim_{n \rightarrow \infty} \frac{p_{n-1}(d\alpha, x)}{p_n(d\alpha, x)} = \rho \left( \frac{x-a}{b} \right).$$

Proof. We have

$$p_{n-1}(d\alpha, x) = \int_{-\infty}^{\infty} p_{n-1}(d\alpha, t) K_n(d\alpha, x, t) d\alpha(t).$$

If  $x \in \text{supp}(d\alpha) \setminus [a-b, a+b]$  then by Theorem 3.3.7  $x$  is an isolated point of  $\text{supp}(d\alpha)$ . Hence we can find  $\varepsilon > 0$  such that

$$\begin{aligned} p_{n-1}(d\alpha, x) &= \int_{|x-t| > \varepsilon} p_{n-1}(d\alpha, t) K_n(d\alpha, x, t) d\alpha(t) + \\ &+ \frac{\alpha(x+0) - \alpha(x-0)}{\lambda_n(d\alpha, x)} p_{n-1}(d\alpha, x). \end{aligned}$$

Using

$$K_n(d\alpha, x, t) = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \frac{p_{n-1}(d\alpha, t) p_n(d\alpha, x) - p_n(d\alpha, t) p_{n-1}(d\alpha, x)}{x-t}$$

we obtain

-46-

$$p_{n-1}(d\alpha, x) \left\{ 1 - \frac{\alpha(x+0) - \alpha(x-0)}{\lambda_n(d\alpha, x)} + \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \right\}.$$

$$\int_{|x-t|>\varepsilon} \frac{p_{n-1}(d\alpha, t) Y_n(d\alpha, t)}{x-t} d\alpha(t) =$$

$$= p_n(d\alpha, x) \cdot \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \int_{|x-t|>\varepsilon} \frac{p_{n-1}^2(d\alpha, t)}{x-t} d\alpha(t).$$

We have

$$\lim_{n \rightarrow \infty} \frac{\alpha(x+0) - \alpha(x-0)}{\lambda_n(d\alpha, x)} = 1.$$

(See Freud, Section II.2,  $\text{supp}(d\alpha)$  is compact!) Thus by Theorem 4.2.13

$p_n(d\alpha, x) \neq 0$  for  $n$  large and

$$\lim_{n \rightarrow \infty} \frac{p_{n-1}(d\alpha, x)}{p_n(d\alpha, x)} = \frac{\frac{1}{\pi} \int_{a+b}^{a+b} (x-t)^{-1} [b^2 - (t-a)^2]^{-\frac{1}{2}} dt}{\frac{1}{\pi} \int_{a-b}^{a+b} t(x-t)^{-1} [b^2 - (t-a)^2]^{-\frac{1}{2}} dt}$$

which equals  $\rho\left(\frac{x-a}{b}\right)$ .

Theorem 1'9. Let  $\alpha \in M(0, 1)$  and  $l$  be a fixed nonnegative integer.

Then

$$(II) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+l}(d\alpha, x) = T_l(x)$$

for each  $x \in [-1, 1]$  provided that  $\alpha$  is continuous at  $x$ ; in particular,

(II) holds for almost every  $x \in \text{supp}(d\alpha)$ . If  $\alpha$  is continuous on  $\tau \subset (-1, 1)$

then (II) is satisfied uniformly for  $x \in \tau$ .

Proof. If  $l = 1$  then the theorem follows from Theorem II and from the formula

$$\begin{aligned} x - \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+1}(d\alpha, x) &= \\ &= \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} \alpha_k(d\alpha) p_k^2(d\alpha, x) + \\ &+ \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} \left[ 2 \frac{Y_k(d\alpha)}{Y_{k+1}(d\alpha)} - 1 \right] p_k(d\alpha, x) p_{k+1}(d\alpha, x) - \\ &- \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} p_{n-1}(d\alpha, x) p_n(d\alpha, x) \lambda_n(d\alpha, x) \end{aligned}$$

which is a direct consequence of the recurrence formula. Now let  $l > 1$ .

Then by Theorem 3.1.1

$$\begin{aligned} \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+l}(d\alpha, x) &= \\ &= U_{l-1}(x) \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+l}(d\alpha, x) - \\ &- U_{l-2}(x) + \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) R_{k+l, k+1}(d\alpha, x). \end{aligned}$$

Since  $U_{l-1}(x)x - U_{l-2}(x) = T_l(x)$  we obtain that (II) holds at those points

$x$  where it holds with  $l = 1$  and where

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x) R_{k+l, k+1}(d\alpha, x) = 0$$

is also satisfied. To finish the proof we apply 3.1.(3) and Theorem II.

Recall that

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) = 0$$

at every  $x$  where  $\alpha$  is continuous and the convergence is uniform on every interval of continuity of  $\alpha$  since  $\text{supp}(d\alpha)$  is compact. (See Freud, Section II. 3.)

#### 4.2. Weak Limits

Definition 1. We write  $\alpha \in S$  if  $\text{supp}(d\alpha) = [-1, 1]$  and  $v \log \alpha' \in L^1(-1, 1)$ .

Lemma 2. If  $\alpha \in S$  then

$$\lim_{n \rightarrow \infty} \gamma_n(d\alpha) 2^{-n} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2\pi} \int_{-1}^1 v(t) \log \alpha'(t) dt\right).$$

Proof. See e.g. Freud, §V. 6.

Theorem 3. Let  $\alpha \in S$  and  $f$  be Riemann integrable on  $[-1, 1]$ . Then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{f(x_k)}{1 - x_k^2} = \frac{2}{\pi} \int_{-1}^1 f(t) \frac{dt}{\sqrt{1-t^2}}.$$

Proof. Let  $\beta$  be defined by  $d\beta(x) = (1-x^2) d\alpha(x)$ . Then  $(1-x^2)[p_{n-1}^2(d\beta, x) - \gamma_{n-1}^2(d\beta) x^{2n-2}]$  is a polynomial of degree  $2n-1$  and we have by the

Gauss-Jacobi mechanical quadrature formula

$$\begin{aligned} \sum_{k=1}^n (1-x_k^2) [p_{n-1}^2(d\beta, x_k) - \gamma_{n-1}^2(d\beta) x_k^{2n-2}] \lambda_k &= \\ &= 1 + \gamma_{n-1}^2(d\beta) \int_{-1}^1 (t^2-1) t^{2n-2} d\alpha(t) \end{aligned}$$

(here  $x_k = x_{kn}(d\alpha)$  and  $\lambda_k = \lambda_{kn}(d\alpha)$ ). Thus

$$\begin{aligned} \sum_{k=1}^n (1-x_k^2) p_{n-1}^2(d\beta, x_k) \lambda_k &= 1 + \gamma_{n-1}^2(d\beta) \cdot \\ &\cdot \left[ \int_{-1}^1 t^{2n} d\alpha(t) - \sum_{k=1}^n x_k^{2n} \lambda_k \right]. \end{aligned}$$

Further

$$\sum_{k=1}^n x_k^{2n} \lambda_k = \sum_{k=1}^n L_n(d\alpha, y, x_k) \lambda_k = \int_{-1}^1 L_n(d\alpha, y^{2n}, t) d\alpha(t).$$



Hence we obtain

$$\sum_{k=1}^n (1-x_k^2) p_{n-1}^2(d\beta, x_k) \lambda_k = \\ = 1 + \gamma_{n-1}^2(d\beta) \int_{-1}^1 \{t^{2n} - L_n(d\alpha, y, t)\} d\alpha(t).$$

Since  $t^{2n} - L_n(d\alpha, y, t)$  is a polynomial of degree  $2n$  which vanishes at the zeros of  $p_n(d\alpha, x)$  we have

$$t^{2n} - L_n(d\alpha, y, t) = p_n(d\alpha, t) [\Gamma p_n(d\alpha, x) + \Pi_{n-1}(x)].$$

Comparing the leading coefficients we see that  $\Gamma = \gamma_n(d\alpha)^{-2}$ . Consequently

$$(1) \quad \sum_{k=1}^n (1-x_k^2) p_{n-1}^2(d\beta, x_k) \lambda_k = 1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)}.$$

Expand  $(1-x^2) p_{n-1}(d\beta, x)$  in a Fourier series in  $p_k(d\alpha, x)$ .\*) It is easy to see that

$$(1-x^2) p_{n-1}(d\beta, x) = \sum_{k=n-1}^{n+1} a_k p_k(d\alpha, x)$$

with  $a_{n-1} = \gamma_{n-1}(d\alpha)/\gamma_{n-1}(d\beta)$  and  $a_{n+1} = -\gamma_{n-1}(d\beta)/\gamma_{n+1}(d\alpha)$ . Thus

$$(1-x_k^2) p_{n-1}(d\beta, x_k) \lambda_k = \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} p_{n-1}(d\alpha, x_k) - \\ - \frac{\gamma_{n-1}(d\beta)}{\gamma_{n+1}(d\alpha)} p_{n+1}(d\alpha, x_k).$$

\*) This argument is due to Christoffel and is given in Szegő, Chapter 3. In the following this argument will be used several times.

We obtain from the recursion formula that

$$p_{n+1}(d\alpha, x_k) = - \frac{\gamma_{n-1}(d\alpha) \gamma_{n+1}(d\alpha)}{\gamma_n^2(d\alpha)} p_{n-1}(d\alpha, x_k).$$

Hence

$$(1-x_k^2) p_{n-1}(d\beta, x_k) = \left[ \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta) \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)} \right] \cdot \\ \cdot p_{n-1}(d\alpha, x_k).$$

Putting this into (1) we obtain

$$\sum_{k=1}^n \lambda_k \frac{p_{n-1}^2(d\alpha, x_k)}{1-x_k^2} = \left[ 1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)} \right] \cdot \\ \cdot \left[ \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta) \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)} \right]^{-2}$$

From  $\alpha \in S$  follows  $\beta \in S$  and we can use Lemma 2 to show that the limit of the right hand side is 2. Thus by Theorem 3.2.3 if  $\epsilon \in (0, 1)$  then

$$\lim_{n \rightarrow \infty} \sum_{|x_k| > 1-\epsilon} \lambda_k \frac{p_{n-1}^2(d\alpha, x_k)}{1-x_k^2} = \frac{4}{\pi} \int_{1-\epsilon}^1 v(t) dt.$$

Let  $f$  be Riemann integrable on  $[-1, 1]$ . By Theorem 3.2.3 we have to show that

$$\lim_{\epsilon \rightarrow +0} \limsup_{n \rightarrow \infty} \left| \sum_{|x_k| > 1-\epsilon} \lambda_k \frac{p_{n-1}^2(d\alpha, x_k)}{1-x_k^2} f(x_k) \right| = 0.$$

Since  $f$  must be bounded on  $[-1, 1]$  this holds by the previous formula.

Theorem 3 will be used to investigate some interpolation processes.

Definition 4. Let  $\alpha \in S$ . Then

$$\Gamma(\theta) = -\frac{1}{2\pi} \int_{-1}^1 \frac{\log W(t) - \log W(x)}{t-x} \frac{\sqrt{1-x^2}}{\sqrt{1-t^2}} dt, \quad x = \cos \theta,$$

where  $W(x) = \alpha'(x) \sqrt{1-x^2}$ ,  $-1 < x < 1$ ,  $0 \leq \theta \leq \pi$ .

Lemma 5. If  $\alpha \in S$  then

$$\lim_{n \rightarrow \infty} \int_0^\pi |p_n(d\alpha, \cos \theta) \sqrt{\alpha'(\cos \theta) \sin \theta} - \sqrt{\frac{2}{\pi}} \cos[n\theta - \Gamma(\theta)]^2 d\theta = 0$$

Proof. See Geronimus, Chapter IX.

In the following we shall apply Lemma 5 several times, here we give three applications of it.

Theorem 6. If  $\alpha \in S$  then

$$\lim_{n \rightarrow \infty} \int_{-1}^1 p_n^2(d\alpha, x) d[\alpha_S(x) + \alpha_f(x)] = 0.$$

Proof. By the Riemann-Lebesgue lemma and Lemma 5  $\int_{-1}^1 p_n^2(d\alpha, x) \alpha'_f(x) dx \rightarrow 0$  when  $n \rightarrow \infty$ .

Theorem 7. Let  $\alpha \in S$  and  $f \in L_{d\alpha}^*$ . Then

$$\lim_{n \rightarrow \infty} \int_{-1}^1 f(x) p_n^2(d\alpha, x) d\alpha(x) = \frac{1}{\pi} \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx.$$

Proof. Use Lemma 5, Theorem 6 and the Riemann-Lebesgue lemma.

Theorem 8. Let  $\alpha \in S$ ,  $0 < p < \infty$ ,  $g(\geq 0) \in L_{d\alpha}^1$ . If

$$\liminf_{n \rightarrow \infty} \int_{-1}^1 |p_n(d\alpha, t)|^p g(t) d\alpha(t) = 0$$

then  $g(t) = 0$  for almost every  $t \in [-1, 1]$ .

Proof. Let first  $2 \leq p < \infty$ . Put for  $M > 0$

$$q_M(t) = \min\{g(t), M\}.$$

Then  $q_M \in L_{d\alpha}^\infty$ . Further

$$\int_{-1}^1 p_n^2(d\alpha, t) q_M(t)^p d\alpha(t) \leq \left[ \int_{-1}^1 |p_n(d\alpha, t)|^p g(t) d\alpha(t) \right]^{\frac{2}{p}} \cdot \frac{p-2}{[\alpha(1) - \alpha(-1)]^{\frac{p}{p-2}}}.$$

By the hypothesis and Theorem 7

$$\frac{1}{\pi} \int_{-1}^1 q_M(t)^{\frac{2}{p}} \frac{dt}{\sqrt{1-t^2}} = 0$$

for every  $M > 0$ . Hence  $g = 0$ . Let now  $1 \leq p < 2$ . Let  $g^*(t) = g(\cos t)$ ,  $\varepsilon > 0$ . Then

$$\begin{aligned} \int_{-1}^1 |p_n(d\alpha, t)|^p g(t) d\alpha(t) &\geq \int_{-1}^1 |p_n(d\alpha, t)|^p g(t) \alpha'(t) dt = \\ &= \int_0^\pi |p_n(d\alpha, \cos t) \sqrt{\alpha'(\cos t) \sin t}|^p |\alpha'(\cos t) \sin t|^{1-\frac{p}{2}} g(t) dt. \end{aligned}$$

Let  $q_1(t) = [\alpha'(\cos t) \sin t]^{1-\frac{p}{2}} g(t)$ . Then

$$\begin{aligned}
& \left| \int_{-1}^1 |p_n(d\alpha, t)|^p |g(t) d\alpha(t)|^{\frac{1}{p}} \right| \geq \\
& \geq \left\{ \varepsilon \int_{g_1 \geq \varepsilon} |p_n(d\alpha, \cos t) \sqrt{\sigma'(\cos t) \sin t}|^p dt \right\}^{\frac{1}{p}} \geq \\
& \geq \varepsilon^{\frac{1}{p}} \left\{ \int_{g_1 \geq \varepsilon} \left| \sqrt{\frac{2}{\pi}} \cos(nt - \Gamma(t)) \right|^p dt \right\}^{\frac{1}{p}} - \\
& - \varepsilon^{\frac{1}{p}} \left\{ \int_0^{\pi} |p_n(d\alpha, \cos t) \sqrt{\sigma'(\cos t) \sin t}|^p dt \right\}^{\frac{1}{p}}.
\end{aligned}$$

Since  $p < 2$ ,  $|\cos|^p \geq |\cos|^2$  and the second integral here converges to 0 by Lemma 5. Thus by the hypothesis

$$\liminf_{n \rightarrow \infty} \int_{g_1 \geq \varepsilon} \cos^2(nt - \Gamma(t)) dt = 0,$$

that is  $\text{mes}\{g_1 \geq \varepsilon\} = 0$ . Thus  $g_1 = 0$  and consequently  $g = 0$ . If

$0 < p < 1$  we can repeat the previous arguments, the only difference is that we consider  $\int |||^p$  instead of  $\int |||^{\frac{1}{p}}$ .

Lemma 9. Let  $\alpha \in M(a, 0)$ . Let  $\{n_k\}$  and  $\{m_k\}$  be two sequences of natural integers such that at least one of them converges to  $\infty$  when  $k \rightarrow \infty$ .

If  $f$  is continuous on  $\Delta(d\alpha)$  then

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f(t) p_{n_k}(d\alpha, t) p_{m_k}(d\alpha, t) d\alpha(t) = \begin{cases} f(a) & \text{if } \lim_{k \rightarrow \infty} (n_k - m_k) = 0 \\ 0 & \text{if } \liminf_{k \rightarrow \infty} |n_k - m_k| > 0. \end{cases}$$

Proof. In the first case we can suppose without loss of generality that

$n_k - m_k = k$  for every  $k$ . Because of continuity we can also suppose that

$f$  is a polynomial. If  $f$  is constant the Lemma is certainly true. Otherwise

$$f(x) = f(a) + \sum_{j=1}^{\deg f} \frac{f^{(j)}(a)}{j!} (x-a)^j.$$

We shall show that for every  $j \geq 1$

$$(2) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} (x-a)^j p_k^2(d\alpha, x) d\alpha(x) = 0.$$

If  $j = 1$  then (2) means that  $\lim_{k \rightarrow \infty} \alpha_k(d\alpha) = a$ . If  $j = 2$  then

$$\begin{aligned}
(x-a)^2 p_k^2(d\alpha, x) &= (x-a) p_k(d\alpha, x) \left[ \frac{Y_k(d\alpha)}{Y_{k+1}(d\alpha)} p_{k+1}(d\alpha, x) + \right. \\
&\quad \left. + (\alpha_k(d\alpha) - a) p_k(d\alpha, x) + \frac{Y_{k-1}(d\alpha)}{Y_k(d\alpha)} p_{k-1}(d\alpha, x) \right].
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} (x-a)^2 p_k^2(d\alpha, x) d\alpha(x) = \frac{Y_k^2}{Y_{k+1}^2} + (\alpha_k - a)^2 + \frac{Y_{k-1}^2}{Y_k^2} \xrightarrow{k \rightarrow \infty} 0.$$

Since  $\text{supp}(d\alpha)$  is compact (2) holds also for  $j > 2$  if it holds for  $j = 2$ .

The second case can be obtained from the first one as follows. If  $k$  is large then  $m_k \neq n_k$ . Thus

$$\begin{aligned}
& \int_{-\infty}^{\infty} f(t) p_{n_k}(d\alpha, t) p_{m_k}(d\alpha, t) d\alpha(t) = \\
& = \int_{-\infty}^{\infty} [f(t) - f(a)] p_{m_k}(d\alpha, t) p_{n_k}(d\alpha, t) d\alpha(t),
\end{aligned}$$

that is the absolute value of the left side is not greater than

$$\left\{ \int_{-\infty}^{\infty} |f(t) - f(a)| p_{n_k}^2(d\alpha, t) d\alpha(t) \right\}^{\frac{1}{2}} \cdot \left\{ \int_{-\infty}^{\infty} |f(t) - f(a)| p_{m_k}^2(d\alpha, t) d\alpha(t) \right\}^{\frac{1}{2}}.$$

Here both factors are bounded and at least one of them tends to 0 when  $k \rightarrow \infty$ .

Theorem 10. Lemma 9 remains true if  $f$ , instead of being continuous on  $\Delta(d\alpha)$ , is merely bounded on  $\text{supp}(d\alpha)$ , continuous at  $a$  and it is  $d\alpha$  measurable.

Proof. Let  $\varepsilon > 0$ . Then

$$\int_{-\infty}^{\infty} q(t) p_k^2(d\alpha, t) d\alpha(t) \leq \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t) d\alpha(t) \leq 1$$

where  $q$  is continuous function vanishing outside  $[a-\varepsilon, a+\varepsilon]$  with

$q(a) = 1$  and  $0 < q(t) \leq 1$  for  $|a-t| \leq \varepsilon$ . Thus by Lemma 9

$$\lim_{k \rightarrow \infty} \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t) d\alpha(t) = 1.$$

We have

$$\left| \int_{-\infty}^{\infty} f(t) p_k^2(d\alpha, t) d\alpha(t) - f(a) \right| \leq \sup_{|t-a| \leq \varepsilon} |f(t) - f(a)| \cdot \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t) d\alpha(t) + 2 \sup_{t \in \text{supp}(d\alpha)} |f(t)| \left[ 1 - \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t) d\alpha(t) \right].$$

Let  $k \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ . The case when  $\lim_{n \rightarrow \infty} \inf |m_k - n_k| > 0$  follows from the case when  $\lim_{k \rightarrow \infty} (m_k - n_k) = 0$ .

Definition 11. Let us define the numbers  $\alpha_{nk}(d\alpha)$  for  $n = 1, 2, \dots$  and

$k = n-1, n, n+1$  as follows

$$\alpha_{nk}(d\alpha) = \begin{cases} \gamma_{n-1}(d\alpha)/\gamma_n(d\alpha) & \text{for } k = n-1 \\ \alpha_n(d\alpha) & \text{for } k = n \\ \gamma_n(d\alpha)/\gamma_{n+1}(d\alpha) & \text{for } k = n+1. \end{cases}$$

Lemma 12. Let  $m$  be a nonnegative integer and  $n > m$ . Then

$$x^m p_n(d\alpha, x) = \sum_{-1 \leq k_1 \leq 1} \alpha_{n, n+k_1}(d\alpha) \alpha_{n+k_1, n+k_1+k_2}(d\alpha) \cdot \dots \alpha_{n+k_1+\dots+k_m, n+k_1+\dots+k_m}(d\alpha, x).$$

$$i=1, 2, \dots, m$$

Proof. Apply the recursion formula repeatedly.

Theorem 13. Let  $\alpha \in M(a, b)$  with  $b > a$ . Let  $\{m_k\}$  and  $\{n_k\}$  be two sequences of natural integers such that at least one of them converges to  $\infty$  when  $k \rightarrow \infty$  and the finite or infinite  $\lim_{k \rightarrow \infty} (m_k - n_k)$  exists. Let  $f$  be  $d\alpha$  measurable, bounded on  $\text{supp}(d\alpha)$  and continuous on  $[a, b]$ .

Then

$$(3) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f(t) p_{m_k}(d\alpha, t) p_{n_k}(d\alpha, t) d\alpha(t) = \lim_{k \rightarrow \infty} \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{T_{|m_k - n_k|}(\frac{t-a}{b})}{\sqrt{b^2 - (t-a)^2}} dt.$$



Proof. Let, without loss of generality,  $\alpha \in M(0, 1)$ . First we shall prove (3) when  $f$  is continuous on  $\Delta(d\alpha)$ . If  $\lim_{k \rightarrow \infty} (m_k - n_k) = \infty$  then the right side in (3) equals 0. If  $f$  is a polynomial then the integral on the left side of (3) equals 0 if  $k$  is big. Thus for every continuous function  $f$  the left side in (3) also equals 0. Now let  $\lim_{k \rightarrow \infty} (m_k - n_k) < \infty$ . Then we may assume that for every  $k$   $n_k = k$ ,  $m_k = k + l$  where  $l$  is a fixed nonnegative integer. Because of linearity and continuity arguments; that is, because of Banach-Steinhaus' theorem, we can suppose that  $f$  is of the form  $f(t) = t^m$  where  $m$  is a fixed nonnegative integer. Thus we have to show that for  $m = 0, 1, 2, \dots$

$$(4) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} t^m p_k(d\alpha, t) p_{k+l}(d\alpha, t) d\alpha(t) = \\ = \frac{1}{\pi} \int_{-1}^1 t^m \frac{T_l(t)}{\sqrt{1-t^2}} dt.$$

Let us remark that (4) is true if  $\alpha$  is the Tschebyshev weight. For if  $k \geq 1$

$$\int_{-\infty}^{\infty} t^m p_k(d\alpha, t) p_{k+l}(d\alpha, t) d\alpha(t) = \\ = \frac{2}{\pi} \int_0^{\pi} \cos^m \theta \cdot \frac{1}{2} [\cos l \theta + \cos(2k+l)\theta] d\theta$$

which equals to

$$(5) \quad \frac{1}{\pi} \int_{-1}^1 t^m \frac{T_l(t)}{\sqrt{1-t^2}} dt$$

if  $2k + l > m$ . If  $\alpha \in M(0, 1)$  then by Lemma 12 we have for  $k > m$

$$\int_{-\infty}^{\infty} t^m p_k(d\alpha, t) p_{k+l}(d\alpha, t) d\alpha(t) = \\ = \sum_{i=1}^m \sum_{j=1}^m \alpha_{k+k_1, k+k_1+k_2}^{(d\alpha)} \alpha_{k+k_1, k+k_1+k_2}^{(d\alpha)} \dots \alpha_{k+k_1, k+k_1+k_2}^{(d\alpha)} \\ i=1, 2, \dots, m \\ \sum_{j=1}^m k_j = l$$

The right side here is convergent when  $k \rightarrow \infty$  since  $\alpha \in M(0, 1)$  and  $l$  is fixed, its limit depends only on  $\lim_{j \rightarrow \infty} \alpha_j(d\alpha)$  and  $\lim_{j \rightarrow \infty} \gamma_{j-1}(d\alpha)/\gamma_j(d\alpha)$ , that is in our case this limit equals (5). Hence (4) holds if  $f$  is continuous on  $\Delta(d\alpha)$ . If  $f$  is continuous only on  $\text{supp}(d\alpha)$  which is closed then  $f$  can be extended to a function which is continuous on  $\Delta(d\alpha)$ . If  $f$  is a function satisfying the conditions of the theorem then we can write  $f = f_1 + f_2$  where  $f_1$  is continuous on  $\Delta(d\alpha)$ ,  $f_2$  is bounded and  $d\alpha$  measurable, further  $f_2$  vanishes on  $[a-b, a+b]$ . Hence, if we can show that

$$(6) \quad \lim_{k \rightarrow \infty} \int_{a+b}^{\infty} p_k^2(d\alpha, t) d\alpha(t) = 0$$

then we finish the proof of the theorem. Let  $g$  be continuous function on  $\Delta(d\alpha)$  such that  $g(t) \geq 0$  for  $t \in \Delta(d\alpha)$ ,  $g(t) = 1$  for  $t \in \Delta(d\alpha) \setminus [a-b, a+b]$  and

$$\frac{1}{\pi} \int_{a-b}^{a+b} g(t) \frac{1}{\sqrt{b^2 - (t-a)^2}} dt < \epsilon$$

where  $\epsilon > 0$  is given. Then

$$\int_{-\infty}^{a-b} \int_{a+b}^{\infty} p_k^2(d\alpha, t) d\alpha(t) \leq \int_{-\infty}^{\infty} q(t) p_k^2(d\alpha, t) d\alpha(t)$$

and using the fact that the theorem has already been proved for continuous functions we obtain (6) by first letting  $k \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ .

Using one-sided approximation machinery we obtain immediately from Theorem 13 the following

Theorem 14. Let  $\alpha \in M(a, b)$  with  $b > 0$ . If  $f$  is  $d\alpha$  measurable, bounded on  $\text{supp}(d\alpha)$  and Riemann integrable on  $[a-b, a+b]$  then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(t) p_n^2(d\alpha, t) d\alpha(t) = \frac{1}{\pi} \int_{a-b}^{a+b} \frac{f(t) dt}{\sqrt{b^2 - (t-a)^2}}$$

Compares this theorem with Theorem 7.

Corollary 15. Let  $\alpha \in M(a, b)$ ,  $b > 0$ ,  $x \in (a-b, a+b)$  and let  $\alpha$  be continuous in a neighborhood of  $x$ . Then

$$\lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{2h} \int_{x-h}^{x+h} p_n^2(d\alpha, t) d\alpha(t) = \frac{1}{\pi} \sqrt{\frac{1}{b^2 - (t-a)^2}}$$

Proof. The function  $1_{(x-\epsilon, x+\epsilon)}$  is  $d\alpha$  measurable for  $\epsilon > 0$  small.

Theorem 16. Let  $\alpha \in M(a, b)$ ,  $b > 0$ . Let  $f$  be bounded on  $\Delta(d\alpha)$  and Riemann integrable on  $[a-b, a+b]$ . Then for every fixed integer  $l$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) p_{n-l}(d\alpha, x_{kn}) p_{n+l}(d\alpha, x_{kn}) &= \\ &= -\text{sign } l \frac{2}{\pi b} \int_{a-b}^{a+b} f(t) U_l \left| \frac{t-a}{b} \right| \sqrt{b^2 - (t-a)^2} dt. \end{aligned}$$

Proof. We obtain from Theorems 3.1.3 ( $l > 0$ ) and 3.1.13 ( $l < 0$ ) and from the recurrence formula that

$$\begin{aligned} p_{n+l}(d\alpha, x_{kn}) &= -\text{sign } l U_l \left| \frac{x_{kn}-a}{b} \right| p_{n-l}(d\alpha, x_{kn}) \\ &\quad + \sigma(l) (|p_{n-l}(d\alpha, x_{kn})| + |p_{n-2}(d\alpha, x_{kn})|) \end{aligned}$$

where  $\lim_{n \rightarrow \infty} \sigma(l) = 0$  uniformly for  $1 \leq k \leq n$  if  $l$  is fixed. Thus the theorem follows from Theorem 3.2.3.

Theorem 17. Let  $\alpha \in S$ ,  $f$  be Riemann integrable on  $[-1, 1]$  and  $l$  be a fixed integer. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) p_{n-l}(d\alpha, x_{kn}) \frac{p_{n+l}(d\alpha, x_{kn})}{1-x_{kn}^2} &= \\ &= -\text{sign } l \frac{1}{\pi} \int_{-1}^1 f(t) U_l \left| \frac{t}{1-t^2} \right| \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Proof. Repeat the proof of Theorem 16 and use Theorem 3 instead of Theorem 3.2.3.

## 5. Eigenvalues of Toeplitz Matrices

In Grenander-Szegő the proof of Theorem 7.7 and the example 8.1(f) are not correct. In the first the Gauss-Jacobi mechanical quadrature formula is used for polynomials of degree more than  $2n-1$ , in the second an orthonormal system is constructed, but it is, in fact, only normed but not orthogonal. In this section we shall obtain results which are a little bit more general than those of Grenander-Szegő, we shall use some methods of the above book in a simplified form.

Lemma 1. Let  $\text{supp}(d\alpha)$  be compact and let  $f$  be continuous on  $\Delta(d\alpha)$  with the modulus of continuity  $\omega$ . Then

$$\begin{aligned} & \left| \sum_{k=1}^n f(x_{kn}) - \sum_{k=0}^{n-1} \int_0^\infty f(t) p_k^2(d\alpha, t) d\alpha(t) \right| \leq \\ & \leq n \omega(n^{-1/3}) \left[ 1 + \frac{1}{2} |\Delta(d\alpha)|^3 \right] \end{aligned}$$

for  $n > |\Delta(d\alpha)|^{-3}$ .

Proof. Since  $\sum_{k=0}^{n-1} p_k^2(t) = \sum_{k=1}^n \frac{f_k^2(t)}{\lambda_k}$  and  $\lambda_k = \int_{-\infty}^\infty t^2 p_k^2(t) d\alpha(t)$  we have

$$\begin{aligned} & \sum_{k=1}^n f(x_{kn}) - \sum_{k=0}^{n-1} \int_0^\infty f(t) p_k^2(d\alpha, t) d\alpha(t) = \\ & = \sum_{k=1}^n \int_{\Delta(d\alpha)} [f(x_{kn}) - f(t)] \frac{f_k^2(d\alpha, t)}{\lambda_{kn}} d\alpha(t) \equiv A. \end{aligned}$$

Let  $\varepsilon > 0$ . Then

$$\begin{aligned} |A| & \leq n \omega(\varepsilon) + \sum_{k=1}^n \int_{|t-x_k| \geq \varepsilon} |f(x_k) - f(t)| \frac{f_k^2(t)}{\lambda_k} d\alpha(t) \leq \\ & \leq n \omega(\varepsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2} \frac{\omega(|\Delta(d\alpha)|)}{\varepsilon} \sum_{k=1}^n \lambda_k p_{n-1}^2(x_k) \int_{-\infty}^\infty t^2 p_n^2(t) d\alpha(t) = \\ & = n[\omega(\varepsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2} \frac{\omega(|\Delta(d\alpha)|)}{n \varepsilon^2}]. \end{aligned}$$

By easy calculation  $\gamma_{n-1}/\gamma_n \leq |\Delta(d\alpha)|/2$ . Thus

$$|A| \leq n[\omega(\varepsilon) + \frac{1}{4} |\Delta(d\alpha)|^2 \frac{\omega(|\Delta(d\alpha)|)}{n \varepsilon^2}].$$

Let now  $\varepsilon = n^{-1/3}$  and use the inequality

$$\frac{\omega(a)}{a} \leq 2 \frac{\omega(b)}{b} \quad (0 < b \leq a)$$

which holds for every modulus of continuity  $\omega$ .

Theorem 2. Let  $\alpha \in M(a, 0)$ . Let  $f$  be bounded on  $\Delta(d\alpha)$  and continuous at  $a$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(x_{kn}(d\alpha)) = f(a).$$

Proof. If  $f$  is continuous on  $\Delta(d\alpha)$  then the theorem follows immediately from Lemma 1 and Lemma 4.2.9. If  $f$  is continuous only at  $a$  and it is bounded on  $\Delta(d\alpha)$  then we fix  $\varepsilon > 0$  and we construct two continuous functions  $f_1$  and  $f_2$  on  $\Delta(d\alpha)$  so that

$$f_1(x) \leq f(x) \leq f_2(x)$$

for  $x \in \Delta(d\omega)$  and  $f_2(a) - f_1(a) \leq \varepsilon$  and we apply standard arguments.

**Theorem 3.** Let  $\alpha \in M(a, b)$  with  $b > 0$ . Let  $f$  be bounded on  $\Delta(d\omega)$  and Riemann integrable on  $[a-b, a+b]$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(x_{kn}(d\omega)) = \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{1}{\sqrt{b^2 - (t-a)^2}} dt,$$

in particular, for every segment  $\Delta \subset \Delta(d\omega)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_{kn}(d\omega) \Delta = \frac{1}{\pi} \int_{a-b}^{a+b} \Delta \cap [a-b, a+b] \sqrt{b^2 - (t-a)^2} dt.$$

**Proof.** If  $f$  is continuous on  $\Delta(d\omega)$  then use Lemma 1 and Theorem 4.2.13, otherwise apply the one-sided approximation machinery.

Now we shall translate the previous results into a different language. Let  $\text{supp}(d\omega)$  be compact,  $f \in L^1_{d\omega}$  be real valued and let us consider the Hilbert matrix  $A(f, d\omega)$  defined as

$$A(f, d\omega) = \left[ \int_{-\infty}^{\infty} f(t) p_i(d\omega, t) p_j(d\omega, t) d\omega(t) \right]_{i,j=0}^{\infty}.$$

Let, further,  $A_n(f, d\omega)$  be the truncated matrix consisting of  $n^2$  elements.

The characteristic polynomial  $h_n(f, d\omega, x)$  is  $\det[A_n(f, d\omega) - xE]$ , the zeros of  $h_n(f, d\omega, x)$ , which we denote by  $x_{kn}(f, d\omega)$  ( $k = 1, 2, \dots, n$ ) are called the eigenvalues of  $A_n(f, d\omega)$ . Since  $A_n^* = A_n$  all  $x_{kn}$  are real.

If  $f(t) \equiv 1$  then  $A(f, d\omega) = E$  and  $h_n(f, d\omega, x) = (1-x)^n$ , that is  $x_{kn}(f, d\omega) = 1$  for  $k = 1, 2, \dots, n$ .

**Lemma 4.** Let  $f(t) \equiv 1$ . Then for  $n = 1, 2, \dots$

$$h_n(f, d\omega, x) = (-1)^{n-1} \gamma_n^{-1}(d\omega) p_n(d\omega, x).$$

**Proof.**  $(-1)^{n-1} h_n(f, d\omega, x)$  satisfy the same recurrence formula as  $\gamma_n^{-1}(d\omega) p_n(d\omega, x)$ , and for  $n = 1, 2$  the lemma can easily be checked.

**Definition 5.**  $\{a_{kn}\}_{k=1}^n$  and  $\{b_{kn}\}_{k=1}^n$  ( $n = 1, 2, \dots$ ;  $a_{kn} \in \mathbb{R}$ ,  $b_{kn} \in \mathbb{R}$ ) are equally distributed if there exists an interval  $\Delta$  such that  $a_{kn} \in \Delta$  and  $b_{kn} \in \Delta$  for  $n = 1, 2, \dots$  and  $k = 1, 2, \dots, n$ , further for every continuous function  $f$  on  $\Delta$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \{f(a_{kn}) - f(b_{kn})\} = 0.$$

We obtain from Theorems 2 and 3 the following

**Theorem 6.** Let  $f(t) \equiv t$ ,  $a \in \mathbb{R}$ ,  $b \geq 0$ . Then for every pair of weights  $\alpha_1$  and  $\alpha_2$  from  $M(a, b)$  the eigenvalues of  $A_n(f, d\alpha_1)$  and  $A_n(f, d\alpha_2)$  are equally distributed.

**Definition 7.** Let  $A = [a_{ik}]_{i,k=1}^n$  be a real  $n \times n$  matrix. Then

$$\text{Tr} A = \sum_{k=1}^n a_{kk},$$

$$\|A\| = \left| \frac{1}{n} \text{Tr} A \right|,$$

$$((A)) = \sup_u \frac{(Au, Au)}{(u, u)},$$

here  $u = (u_1, \dots, u_n)$ ,  $(u, v) = \sum_{k=1}^n u_k v_k$ , further



Proof. For  $m = 1$  the lemma is certainly true. Let  $m \geq 2$ . By Properties 8 we can suppose that  $j = 1$ . Let

$$B = \prod_{i=2}^m (f_i, d\alpha).$$

Then

$$\begin{aligned} \|A_n(f, d\alpha) B\| &= \left\| \frac{1}{n} \sum_{k=1}^n a_{kj}(f, d\alpha) b_{jk} \right\| \leq \\ &\leq \left\| \frac{1}{n} \sum_{k=1}^n \left( \sum_{j=1}^n a_{kj}(f, d\alpha) \sum_{j=1}^n b_{jk}^2 \right)^{\frac{1}{2}} \right\| \leq \\ &\leq \max_{k=1, 2, \dots, n} \left( \sum_{j=1}^n b_{jk}^2 \right)^{\frac{1}{2}} \left\| \frac{1}{n} \sum_{k=1}^n a_{kj}^2(f, d\alpha) \right\|^{\frac{1}{2}}. \end{aligned}$$

By Bessel's inequality

$$\begin{aligned} \sum_{j=1}^n a_{kj}^2(f, d\alpha) &= \sum_{j=0}^{n-1} \left( \int_{-\infty}^{\infty} p_{k-1}(d\alpha, t) p_j(d\alpha, t) f_1(t) d\alpha(t) \right)^2 \leq \\ &\leq \int_{-\infty}^{\infty} p_{k-1}^2(d\alpha, t) f_1^2(t) d\alpha(t) = a_{kk}^2(f_1^2, d\alpha). \end{aligned}$$

Thus by Properties 8

$$\begin{aligned} \|A_n(f_1, d\alpha) B\| &\leq \| \sqrt{A_n(f_1^2, d\alpha)} \| \|(B)\| \leq \\ &\leq \| \sqrt{A_n(f_1, d\alpha)} \| \prod_{i=2}^m \|(A_n(f_i, d\alpha))\| \end{aligned}$$

and now use Lemma 9.

Lemma 11. Let  $\text{supp}(d\alpha)$  be compact,  $s \in \mathbb{N}$  and  $\pi$  be a polynomial. Then

-68-

$$\sqrt{A} = \{(\sqrt{a_{ik}})\}_{i,k=1}^n.$$

Properties 8.  $\text{Tr } AB = \text{Tr } BA$ ,  $\text{Tr } A = \Sigma(\text{eigenvalues of } A)$ ,

$$((A))^2 \geq \max_{k=1, 2, \dots, n} \sum_{j=1}^n a_{kj}^2, \quad ((AB)) \leq ((A))((B)).$$

If  $A^* = A$  then  $((A)) = \max |\text{eigenvalues of } A|$ .

Lemma 9. For every  $n \in \mathbb{N}^+$

$$((A_n(f, d\alpha))) \leq \sup |f(t)| \cdot \text{tr } \text{supp}(d\alpha).$$

Proof. Let  $\lambda$  be an eigenvalue of  $A_n(f, d\alpha)$  for which  $((A_n(f, d\alpha))) = |\lambda|$  and let  $u$  be the corresponding eigenvector with  $(u, u) = 1$ . Then

$$\begin{aligned} ((A_n(f, d\alpha))) &= |\lambda| (u, u) = |(Au, u)| = |(A_n(f, d\alpha)u, u)| = \\ &= \left| \int_{-\infty}^{\infty} f(t) \sum_{k=0}^{n-1} p_k(d\alpha, t) u_k \right|^2 d\alpha(t) \leq \sup |f(t)| (u, u) \text{tr } \text{supp}(d\alpha) \end{aligned}$$

where  $u = (u_0, u_1, \dots, u_{n-1})$ .

Lemma 10. Let  $m \in \mathbb{N}$  be fixed and let  $f_i$  ( $i = 1, 2, \dots, m$ ) be given. Then for every  $j \in \{1, 2, \dots, m\}$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| \prod_{i=1}^m A_n(f_i, d\alpha) \right\| &\leq \\ &\leq \limsup_{n \rightarrow \infty} \left\| \sqrt{A_n(f_j^2, d\alpha)} \right\| \prod_{i=1, i \neq j}^m \sup_{t \in \text{supp}(d\alpha)} |f_i(t)|. \end{aligned}$$

-67-

$$\lim_{n \rightarrow \infty} \|A_n^S(\pi, d\alpha) - A_n^S(\pi^S, d\alpha)\| = 0.$$

Proof. See Grenander-Szegő, §8.1.

Let us remark that in the previous lemma it is sufficient to suppose that for  $d\alpha$  the moment problem is well defined, that is for  $x^k f$ ,  $\int x^k g \in L^2 d\alpha$  ( $k = 0, 1, \dots$ )  $A(fg, d\alpha) = A(f, d\alpha) A(g, d\alpha)$ .

Lemma 12. Let  $\alpha \in M(a, b)$ ,  $f$  be  $d\alpha$  measurable and bounded on  $\text{supp}(d\alpha)$ . If  $b = 0$  and  $f$  is continuous at  $a$  then

$$\lim_{n \rightarrow \infty} \|\sqrt{A_n(f^2, d\alpha)}\| = |f(a)|.$$

If  $b > 0$  and  $f$  is Riemann integrable on  $[a-b, a+b]$  then

$$\lim_{n \rightarrow \infty} \|\sqrt{A_n(f^2, d\alpha)}\| = \left[ \frac{1}{\pi} \int_{a-b}^{a+b} f^2(t) \frac{dt}{\sqrt{b^2 - (t-a)^2}} \right]^{\frac{1}{2}}.$$

Proof. See Theorems 4.2.10 and 4.2.14.

Let us recall that we consider Toeplitz matrices  $A(f, d\alpha)$  for real valued functions  $f$ .

Theorem 13. Let  $\alpha \in M(a, b)$ ,  $s \in \mathbb{N}$ ,  $f$  be  $d\alpha$  measurable and bounded on  $\text{supp}(d\alpha)$ . Let for  $b = 0$   $f$  be continuous at  $a$  and for  $b > 0$   $f$  be Riemann integrable on  $[a-b, a+b]$ . Then

$$\lim_{n \rightarrow \infty} \|A_n^S(f, d\alpha) - A_n^S(f^S, d\alpha)\| = 0.$$

Proof. Let  $\pi$  be a polynomial. Since  $\text{Tr } AB = \text{Tr } BA$  we have

$$\begin{aligned} \|A_n^S(f, d\alpha) - A_n^S(\pi^S, d\alpha)\| &= \|A_n^S(f - \pi + \pi, d\alpha) - \\ &- A_n^S((f - \pi + \pi)^S, d\alpha)\| = \\ &= \|[A_n^S(\pi, d\alpha) - A_n^S(\pi^S, d\alpha)] + \\ &+ \sum_{j=1}^s A_n^j(f - \pi, d\alpha) A_n^{s-j}(\pi, d\alpha) + \\ &+ A_n(-\sum_{j=1}^s A_n^j(f - \pi)^j \pi^{s-j}, d\alpha)\| = \\ &= \|A_I + A_{II} + A_{III}\|. \end{aligned}$$

Let, for the simplicity,  $b > 0$ . If  $b = 0$  then we shall see from the proof that we can put  $\pi(t) \equiv f(a)$ . By Lemma 11

$$\lim_{n \rightarrow \infty} \|A_I\| = 0.$$

By Theorem 4.2.14

$$(1) \quad \lim_{n \rightarrow \infty} \|A_{III}\| = \left| \frac{1}{\pi} \int_{a-b}^{a+b} \left( \sum_{j=1}^s A_n^j(f(t) - \pi(t))^j \right) \cdot \pi(t)^{s-j} \right| \frac{dt}{\sqrt{b^2 - (t-a)^2}}.$$

We have, further, by Lemma 10

$$\begin{aligned} \lim_{n \rightarrow \infty} \|A_{II}\| &\leq \sum_{j=1}^s \lim_{n \rightarrow \infty} \sup \|\sqrt{A_n((f-\pi)^2, d\alpha)}\| \\ &\cdot \sup_{t \in \text{supp}(d\alpha)} |f(t) - \pi(t)|^{j-1} \sup_{t \in \text{supp}(d\alpha)} |\pi(t)| \end{aligned}$$

Hence by Lemma 12

$$\limsup_{n \rightarrow \infty} \|A_{II}\| \leq \left[ \frac{1}{\pi} \int_{a-b}^{a+b} \frac{|f(t) - \pi(t)|^2}{\sqrt{b^2 - (t-a)^2}} dt \right]^{1/2} \\ + 2^S \sup_{t \in \text{supp}(d\alpha)} (|f(t)| + |\pi(t)|)^{S-1} \equiv R(f, \pi),$$

and from (1) we get the same estimate for  $\lim_{n \rightarrow \infty} \|A_{III}\|$ :

$$\lim_{n \rightarrow \infty} \|A_{III}\| \leq R(f, \pi).$$

The theorem will be proved if we show that for every  $\varepsilon > 0$  one can find a polynomial  $\pi$  such that  $R(f, \pi) < \varepsilon$ . This latter can be shown easily. Let  $f_1$  be a function on  $\Delta(d\alpha)$  such that  $f_1(t) = f(t)$  for  $t \in [a-b, a+b]$ ,  $|f(t)| \leq |f_1(t)|$  for  $t \in \Delta(d\alpha)$  and  $f_1 \in L^{\infty}(\Delta(d\alpha))$ . Let us send  $\Delta(d\alpha)$  to  $[-1, 1]$  by a linear transformation and then to  $[0, \pi]$  by  $x = \cos \theta$  ( $-1 \leq x \leq 1$ ,  $0 \leq \theta \leq \pi$ ). Set  $g(\theta) = f_1(t)$  ( $t \in \Delta(d\alpha)$ ),  $\theta \in [0, \pi]$ . Let  $g^*$  denote the even extension of  $g$  to  $[-\pi, \pi]$ . Consider the Fejér sums of  $g^*$ . They are cosine polynomials, they are bounded in maximum norm:

$$\| \sigma_n(g^*) \| \leq \sup_{t \in \Delta(d\alpha)} |f_1(t)|, \text{ they converge to } g^* \text{ in e.g. } L^1[-\pi, \pi]. \text{ Let us return now to } \Delta(d\alpha) \text{ and remark that}$$

$$\int_{a-b}^{a+b} \frac{|f(t) - \pi(t)|^2}{\sqrt{b^2 - (t-a)^2}} dt \leq C \int_{\Delta(d\alpha)} |f(t) - \pi(t)|^{10} \nu_{\Delta(d\alpha)}(t) dt \Big)^{1/5}.$$

Theorem 14. Let  $\alpha \in M(a, 0)$ ,  $f$  be  $d\alpha$  measurable, bounded on  $\text{supp}(d\alpha)$  and continuous at  $a$ . Let  $\mathfrak{F}$  be bounded on  $\Delta \subset f(\text{supp}(d\alpha))$  and continuous at  $f(a)$ . Then

-71-

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathfrak{F}(x_{kn}(f, d\alpha)) = \mathfrak{F}(f(a)).$$

Proof. Observe that if  $f(\text{supp}(d\alpha)) \subset \Delta$  then  $x_{kn}(f, d\alpha) \in \Delta$  for  $n = 1, 2, \dots$  and  $k = 1, 2, \dots, n$ . This can be shown by exactly the same argument as in Lemma 9. Let first  $\mathfrak{F}(u) = u^s$  ( $s = 0, 1, 2, \dots$ ). For  $s = 0$  the theorem is true. For  $s = 1$

$$\sum_{k=1}^n x_{kn}(f, d\alpha) = \text{Tr } A_n(f, d\alpha)$$

and we apply Theorem 4.2.10. If  $s \geq 2$  then

$$\sum_{k=1}^n x_{kn}^s(f, d\alpha) = \text{Tr } A_n^s(f, d\alpha)$$

as is well known and we use Theorems 13 and 4.2.10. Hence the theorem is true if  $\mathfrak{F}$  is a polynomial, and consequently it is true if  $\mathfrak{F}$  is continuous on  $\Delta$ . Otherwise we use one-sided approximations.

Theorem 15. Let  $\alpha \in M(a, b)$  with  $b > 0$ . Let  $f$  be  $d\alpha$  measurable, bounded on  $\text{supp}(d\alpha)$  and Riemann integrable on  $[a-b, a+b]$ . Let  $\mathfrak{F}$  be continuous on  $\Delta \supset f(\text{supp}(d\alpha))$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathfrak{F}(x_{kn}(f, d\alpha)) = \frac{1}{\pi} \int_{a-b}^{a+b} \frac{\mathfrak{F}(f(t))}{\sqrt{b^2 - (t-a)^2}} dt.$$

Proof. The same as that of Theorem 14.

Theorems 14 and 15 give us the following

Theorem 16. Let  $\alpha \in \mathbb{R}$ ,  $b \geq 0$  and  $f$  be continuous on  $\mathbb{R}$ . Then for each pair of weights  $\alpha_1$  and  $\alpha_2$  belonging to  $M(a, b)$  the eigenvalues of  $A_n(f, d\alpha_1)$  and  $A_n(f, d\alpha_2)$  are equally distributed.

-72-

## 6. Christoffel Functions

### 6.1. An Interpolation Process

The Hermite-Fejér interpolation polynomial  $H_n(d\alpha, f, x)$  is the unique polynomial of degree at most  $2n-1$  which satisfies the conditions

$$H_n(d\alpha, f, x_{kn}) = f(x_{kn}), \quad H'_n(d\alpha, f, x_{kn}) = 0$$

for  $k = 1, 2, \dots, n$ . Here  $x_{kn} = x_{kn}(d\alpha)$ . Hence

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) [1 - 2l'_{kn}(d\alpha, x_{kn})(x - x_{kn})] l_{kn}^2(d\alpha, x).$$

Let us compute  $l'_k(x_k)$ . We have

$$l_k^{-1}(x) = \sum_{k=1}^n \frac{l_k^2(x)}{x_{kn}},$$

that is

$$-l'_k(x) l_k^{-2}(x) = \sum_{k=1}^n \frac{2l'_k(x) l_k(x)}{x_{kn}}.$$

Putting here  $x = x_{kn}$  we obtain

$$(1) \quad -2l'_k(x_k) = l'_k(x_k) l_k^{-1}(x_{kn}).$$

Thus

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) \left[ l_{kn}(d\alpha) + l'_k(d\alpha, x_{kn})(x - x_{kn}) \right] \cdot \frac{l_{kn}^2(d\alpha, x)}{l_{kn}(d\alpha)}.$$

This is Freud's representation for  $H_n(d\alpha, f, x)$ . (See [6]). In the

brackets here we find an approximate expression for  $l_{kn}(d\alpha, x)$ :

$$l_{kn}(x) = l_{kn} + l'_k(x_{kn})(x - x_{kn}) + \frac{l''_k(\theta)}{2} (x - x_{kn})^2$$

where  $\theta$  is between  $x$  and  $x_{kn}$ . Let us replace the expression in the

brackets by  $l_{kn}(d\alpha, x)$ . Denote the resulting expression by  $F_n(d\alpha, f, x)$ :

$$F_n(d\alpha, f, x) = l_{kn}(d\alpha, x) \sum_{k=1}^n \frac{f(x_{kn})}{l_{kn}} \frac{l_{kn}^2(d\alpha, x)}{l_{kn}(d\alpha)}.$$

For  $z \in \mathbb{C}$  put

$$F_n(d\alpha, f, z) = l_{kn}(d\alpha, z) \sum_{k=1}^n \frac{f(x_{kn})}{l_{kn}} \frac{l_{kn}^2(d\alpha, z)}{l_{kn}(d\alpha)}$$

and

$$\bar{F}_n(d\alpha, f, z) = l_{kn}(d\alpha, z) \sum_{k=1}^n \frac{f(x_{kn})}{l_{kn}} \frac{|l_{kn}^2(d\alpha, z)|}{l_{kn}(d\alpha)}.$$

(See 4.1.)

Properties 1. (i) If  $f(x) \equiv 1$  then  $F_n(f, x) \equiv 1$ . (ii) If  $f(x) \geq 0$  for

$x \in \Delta(d\alpha)$  then  $F_n(f, x) \geq 0$  for  $x \in \mathbb{R}$ . (iii)  $F_n(d\alpha, f, x_{kn}) = f(x_{kn})$  for  $k = 1, 2, \dots, n$ . (iv)  $F'_n(d\alpha, f, x_{kn}) = 0$  for  $k = 1, 2, \dots, n$  (use (1)).

(v)  $F_n$  is a rational function of degree  $(2n-2, 2n-2)$ , only the numerator depends on  $f$ .

Because of (ii) we can expect that for many weights  $\alpha$   $F_n(d\alpha, f)$

converges to  $f$  whenever  $f$  is continuous. The surprising result is that



the above class of weights  $\alpha$  is very large. We shall consider convergence of  $F_n(d\alpha, f)$  for  $\alpha \in M(a, b)$  with  $b > 0$  since the case when  $\alpha \in M(a, 0)$  is less interesting. In order to avoid complicated formulas we shall assume, without loss of generality, that  $\alpha \in M(0, 1)$ . Concerning  $\rho(z)$  see Definition 4.1.8.

**Theorem 2.** Let  $\alpha \in M(0, 1)$ . Let  $f$  be bounded on  $\Delta(d\alpha)$ . If  $f$  is continuous at some  $x \in \text{supp}(d\alpha)$  then

$$(2) \quad \lim_{n \rightarrow \infty} F_n(d\alpha, f, x) = f(x).$$

If  $f$  is continuous on the segment  $\Delta \subset (-1, 1)$  then (2) is satisfied uniformly for  $x \in \Delta$ . If  $f$  is Riemann integrable on  $[-1, 1]$  and bounded on  $\Delta(d\alpha)$  then for every  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} F_n(d\alpha, f, z) = \frac{\rho(z)-1}{2\pi} \int_{-1}^1 \frac{f(t) \sqrt{1-t^2}}{(z-t)^2} dt$$

and

$$\lim_{n \rightarrow \infty} \tilde{F}_n(d\alpha, f, z) = \frac{|\rho(z)|^2-1}{2\pi} \int_{-1}^1 \frac{f(t) \sqrt{1-t^2}}{|z-t|^2} dt.$$

**Proof.** The theorem follows from Theorems 3.2.3, 3.3.8, 4.1.11 and Properties 1. Let us prove e.g. the first part of the theorem. Let  $\epsilon > 0$ .

Then

$$|F_n(f, x) - f(x)| \leq \sup_{|x-t| \leq \epsilon} |f(t) - f(x)| + 2 \frac{Y_{n-1}}{2} \epsilon^{-2} \lambda_n(d\alpha, x) p_n^2(d\alpha, x) \sup_{t \in \Delta(d\alpha)} |f(t)|.$$

-75-

First let  $n \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ .

**Definition 3.** Let  $g(\geq 0) \in L_{d\alpha}^1$ . Then  $\alpha_g$  is defined by

$$\alpha_g(t) = \int_{-\infty}^t g(u) d\alpha(u).$$

Let us remark that  $\alpha_g$  may not be a weight, it can happen that either  $\alpha_g$  has only a finite number of points of increase or not each moment of  $\alpha_g$  is finite. If  $g$  is a polynomial then  $\alpha_g$  certainly is a weight. If  $\text{supp}(d\alpha)$  is compact and  $g^{-1} \in L_{d\alpha}^1$  then also  $\alpha_g$  is a weight.

**Lemma 4.** Let  $g$  be a linear function, nonnegative on  $\text{supp}(d\alpha)$  ( $g(t) = c_1 t + c_2$ ,  $c_1 \neq 0$ ). Then

$$(3) \quad \lambda_n^{-1}(d\alpha, x) = \sum_{k=1}^n \frac{f_{kn}^2(d\alpha, x)}{g(x_{kn}(d\alpha)) \lambda_{kn}(d\alpha)}.$$

**Proof.** (By Freud [7]). Let us denote the right hand side of (3) by  $A$ . We have to show that for every  $\pi_{n-1}$

$$(4) \quad \pi_{n-1}^2(x) \leq A \int_{-\infty}^{\infty} \pi_{n-1}^2(t) d\alpha_g(t)$$

and for every  $x \in \mathbb{R}$  there exists a  $\pi_{n-1}^*$  which turns (4) into equality. We have  $\pi_{n-1} \equiv L_n(d\alpha, \pi_{n-1})$ . Hence

$$\pi_{n-1}^2(x) \leq \sum_{k=1}^n \lambda_{kn}(d\alpha) \pi_{n-1}^2(x_{kn}) g(x_{kn}) A.$$

Since  $\deg \pi_{n-1}^2 \leq 2n-1$  we can use the Gauss-Jacobi mechanical quadrature to obtain

-76-

$$\pi_{n-1}^2(x) \leq A \int_{-\infty}^{\infty} \pi_{n-1}^2(t) g(t) d\alpha(t) = A \int_{-\infty}^{\infty} \pi_{n-1}^2(t) d\alpha_g(t).$$

On the other hand we can define  $\pi_{n-1}^*$  by

$$\pi_{n-1}^*(t) = \sum_{k=1}^n \frac{f_{kn}(d\alpha, t) f_{kn}(d\alpha, x)}{g(x_{kn}(d\alpha)) \lambda_{kn}(d\alpha)}.$$

**Lemma 5.** Let  $\text{supp}(d\alpha)$  be compact and let  $c$  be one of the endpoints of  $\Delta(d\alpha)$ . Then

$$\sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{|c - x_{kn}|} \leq \frac{2}{Y_n} \frac{Y_n^2(d\alpha)}{Y_{n-1}(d\alpha)} |c - \alpha_n(d\alpha)|.$$

**Proof.** (By Freud [8]). Since  $p_n(d\alpha, c) \neq 0$

$$\sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{c - x_{kn}} = \frac{Y_n}{Y_{n-1}} \frac{p_{n-1}(d\alpha, c)}{p_n(d\alpha, c)}.$$

Further  $\text{sign } p_{n+1}(d\alpha, c) = \text{sign } p_{n-1}(d\alpha, c)$ . Thus by the recurrence formula

$$|c - \alpha_n| |p_n(d\alpha, c)| \geq \frac{Y_{n-1}}{Y_n} |p_{n-1}(d\alpha, c)|.$$

The lemma follows from the above two formulas.

**Theorem 6.** Let  $\alpha \in M(0, 1)$ . Let  $q(t) = c_1^t + c_2^t$  ( $c_1 \neq 0$ ) be nonnegative on  $\text{supp}(d\alpha)$ . Then for every  $x \in \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, x)}{Y_n(d\alpha, x)} = g(x)$$

and the convergence is uniform for  $x \in \Delta \subset (-1, 1)$ .

-77-

**Proof.** We have by Lemma 4

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha, g^{-1}(x))} = F_n(d\alpha, g^{-1}(x)).$$

If  $g$  is positive on  $\Delta(d\alpha)$  we can directly apply Theorem 2. Next let  $g$  vanish at one of the endpoints of  $\Delta(d\alpha)$  which we denote by  $c$ . First we show that

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, g^{-1}(c))}{\lambda_n(d\alpha, c)} = 0 \quad (= g(c)).$$

Let  $\varepsilon > 0$ . Then  $g + \varepsilon$  is positive on  $\Delta(d\alpha)$ . Hence

$$0 \leq \frac{\lambda_n(d\alpha, g^{-1}(c))}{\lambda_n(d\alpha, c)} \leq \frac{\lambda_n(d\alpha, g + \varepsilon, c)}{\lambda_n(d\alpha, c)} \xrightarrow{n \rightarrow \infty} g(c) + \varepsilon = \varepsilon.$$

Now let  $\varepsilon \rightarrow 0$ . If  $x \in \text{supp}(d\alpha) \setminus c$  then for small  $\delta > 0$   $g^{-1}$  is bounded in  $[x - \delta, x + \delta]$  (and  $g^{-1}$  is uniformly bounded in  $\Delta \subset (-1, 1)$ ). Writing  $g(t) = A(t - c)$  we have

$$\begin{aligned} |F_n(d\alpha, g^{-1}(x)) - g^{-1}(x)| &= \left| \sum_{|x - x_k| \leq \delta} \frac{1}{x - x_k} + \sum_{|x - x_k| > \delta} \frac{1}{x - x_k} \right| \\ &\leq \frac{A\delta}{g(x)} \max_{|t - x| \leq \delta} g^{-1}(t) + \delta^{-2} \frac{Y_{n-1}}{Y_n} \lambda_n(x) p_n^2(x) \\ &\quad + \sum_{|x - x_k| > \delta} |g^{-1}(x) - g^{-1}(x_k)| \lambda_k \frac{p_{n-1}^2(x_k)}{Y_{n-1}} \\ &\leq \frac{A\delta}{g(x)} \max_{|t - x| \leq \delta} g^{-1}(t) + \delta^{-2} \frac{Y_{n-1}}{Y_n} \lambda_n(x) p_n^2(x) q^{-1}(x) + \\ &\quad + A \delta^{-2} \frac{Y_{n-1}}{Y_n} \lambda_n(x) p_n^2(x) \sum_{k=1}^n \lambda_k \frac{p_{n-1}^2(x_k)}{|x_k - c|}. \end{aligned}$$

-78-

By Lemma 5 we obtain

$$|F_n(d\alpha, g^{-1}, x) - g^{-1}(x)| \leq A\delta g^{-1}(x) \max_{|x-t| \leq \delta} g^{-1}(t) + \delta^{-2} \frac{Y_{n-1}}{2} \lambda_n(x) p_n^2(x) g^{-1}(x) + A\delta^{-2} \lambda_n(x) p_n^2(x) |c - \alpha_n|.$$

This estimate and Theorem 4.1.11 shows that

$$\lim_{n \rightarrow \infty} F_n(d\alpha, g^{-1}, x) = g^{-1}(x)$$

for  $x \in \text{supp}(d\alpha) \setminus c$  and the convergence is uniform for  $x \in \Delta \subset (-1, 1)$ .

Lemma 7. Let  $u \in \mathbb{C} \setminus (-1, 1)$  and  $z \in \mathbb{C} \setminus [0, 1]$ . Then

$$\frac{\rho^2(z) - 1}{2\pi} \int_{-1}^1 \frac{\sqrt{1-t^2}}{-1(t-u)(t-z)^2} dt = \frac{1}{z-u} - \frac{\sqrt{z^2-1} \rho(z)}{(z-u)^2} [\rho^{-1}(u) - \rho^{-1}(z)]$$

where  $\sqrt{z^2-1} > 0$  for  $z > 1$ .

Proof. Because of continuity arguments we can suppose that  $z \neq u$  and

$u \in \mathbb{C} \setminus [-1, 1]$ . Since

$$\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-t^2}}{z-t} dt = \frac{1}{\rho(z)}$$

(See the proof of Theorem 4.1.13) we have

$$\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-t^2}}{(u-t)(z-t)} dt = \frac{1}{(z-u)} [\rho^{-1}(u) - \rho^{-1}(z)].$$

Differentiating this identity with respect to  $z$  we obtain the lemma.

Theorem 8. Let  $\alpha \in M(0, 1)$ . Let  $g(t) = A(t-B)$  be positive on  $\Delta(d\alpha)$ . Then for every  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\frac{\lambda_n^*(d\alpha, z)}{\lambda_n^*(d\alpha, z)} = g^{-1}(z) + \sqrt{z^2-1} \frac{d}{dz} g^{-1}(z),$$

$$[\rho^{-1}(B) - \rho^{-1}(z)],$$

in particular,

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, B)}{\lambda_n(d\alpha, g^{-1}B)} = \frac{\rho(B)}{2(B^2-1)A}.$$

Proof. Use Theorem 2, Lemmas 4, 7, Theorem 3.3.8 and Theorem 4.1.11.

Lemma 9. Let  $g(t) = A(t-B)$  ( $A \neq 0$ ) be nonnegative on  $\text{supp}(d\alpha)$ . Then

$$\frac{Y_{n-1}^2(d\alpha)}{Y_{n-1}^2(d\alpha)} = -\frac{1}{A} \frac{Y_n(d\alpha)}{Y_{n-1}(d\alpha)} \frac{p_{n-1}(d\alpha, B)}{p_n(d\alpha, B)}.$$

Proof. We have

$$\frac{Y_{n-1}^2(d\alpha)}{Y_{n-1}^2(d\alpha)} = \lim_{x \rightarrow \infty} \frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha, g^{-1}x)}$$

which equals by Lemma 4

$$\lim_{x \rightarrow \infty} \frac{\sum_{k=1}^n \frac{f_k^2(dx, x) x^2}{\lambda_k(dx) g(x_k)} = \frac{\sum_{k=1}^n \lambda_k(dx) p_{n-1}^2(dx, x_k) g^{-1}(x_k)}{\sum_{k=1}^n \lambda_k(dx) p_{n-1}^2(dx, x_k)} =$$

$$= -\frac{1}{A} \sum_{k=1}^n \lambda_k(dx) p_{n-1}^2(dx, x_k) \frac{1}{B-x_k} =$$

$$= -\frac{1}{A} \frac{Y_n(dx)}{Y_{n-1}(dx)} L_n(dx, p_{n-1}(dx), B) \cdot \frac{1}{p_n(dx, B)} =$$

$$= -\frac{1}{A} \frac{Y_n(dx)}{Y_{n-1}(dx)} \frac{p_{n-1}(dx, B)}{p_n(dx, B)}.$$

Lemma 10. Let  $\alpha \in M(0, 1)$ . Let  $g(t) = A(t-B)$  be positive on  $\Delta(dx)$ .

Then  $\alpha_g \in M(0, 1)$  and

$$(5) \quad \lim_{n \rightarrow \infty} \frac{Y_{n-1}(dx_g)}{Y_{n-1}(dx)} = \left| \frac{2}{A\alpha(B)} \right|^{\frac{1}{2}} = \exp\left(-\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}}\right).$$

Proof. If  $g$  is positive on  $\Delta(dx)$  then  $B$  is outside  $\Delta(dx)$ . Hence by Theorem 4.1.13

$$(6) \quad \lim_{n \rightarrow \infty} \frac{p_{n-1}(dx, B)}{p_n(dx, B)} = \rho^{-1}(B) < \infty.$$

Applying Lemma 9 we see that the equality on the left side of (5) holds and consequently

$$\lim_{n \rightarrow \infty} \frac{Y_{n-1}(dx_g)}{Y_n(dx_g)} = \frac{1}{2}.$$

Putting  $\alpha =$  Tschebyshev weight we have  $\alpha \in S$  and  $\alpha_g \in S$  (Let us recall that  $[-1, 1] \subset \Delta(d\beta)$  for  $\beta \in M(0, 1)$  and hence  $g$  is positive on  $[-1, 1]$ ). Using Lemma 4.2.2 we obtain the right side equality in (5).

Now we have to show that

$$\lim_{n \rightarrow \infty} \alpha_n(dx_g) = 0.$$

Let us develop  $g p_n(dx_g)$  into a Fourier series in  $p_k(dx)$ . It is easy to see that

$$g(x) p_n(dx_g, x) = \frac{Y_n(dx)}{Y_n(dx_g)} p_n(dx, x) + A \frac{Y_n(dx_g)}{Y_{n+1}(dx)} p_{n+1}(dx, x).$$

Hence

$$\int_{-\infty}^{\infty} g^2(x) p_n^2(dx_g, x) dx(x) = \frac{Y_n^2(dx)}{Y_n^2(dx_g)} + A^2 \frac{Y_n^2(dx_g)}{Y_{n+1}^2(dx)}.$$

The left side equals

$$\int_{-\infty}^{\infty} A(x-B) p_n^2(dx_g, x) dx_g(x) = \Lambda_{\alpha_g}(dx_g) - AB.$$

Thus by Lemma 9

$$\alpha_n(dx_g) = B - \frac{Y_n(dx)}{Y_{n+1}(dx)} \left\{ \frac{p_{n+1}(dx, B)}{p_n(dx, B)} + \frac{p_n(dx, B)}{p_{n+1}(dx, B)} \right\}.$$

By (6)  $\lim_{n \rightarrow \infty} \alpha_n(dx_g)$  exists and equals  $B - \frac{1}{2} [\rho(B) + \rho^{-1}(B)] = 0$ . Consequently  $\alpha_g \in M(0, 1)$ .



Remark 11. Lemma 9 and the proof of Lemma 10 show that if  $g(t) = A(t-B)$  is positive on  $\Delta(d\alpha)$  and  $\alpha \in M(0,1)$  then

$$\lim_{n \rightarrow \infty} \frac{g(z) p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{A}{\sqrt{2}} \left| \frac{1}{A\alpha(B)} \right|^{\frac{1}{2}} [\rho(z) - \rho(B)]$$

for  $z \in \mathbb{C} \setminus \text{supp}(d\alpha) \setminus \{B\} = \mathbb{C} \setminus \text{supp}(d\alpha_g) \setminus \{B\}$ .

This remark and Theorem 4.1.11 give a new proof of Theorem 8.

Lemma 12. Let  $\alpha \in M(0,1)$ . Let  $g(x) = (x-A)^2 + B^2$  with  $A \in \mathbb{R}$ ,  $B^2 > 0$ . Then  $\alpha_g \in M(0,1)$  and

$$(7) \quad \lim_{n \rightarrow \infty} \frac{Y_n(d\alpha_g)}{Y_n(d\alpha)} = 2 \left| \frac{1}{\rho(A+1B)\rho(A-1B)} \right|^{\frac{1}{2}} = \exp \left( -\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right).$$

Proof. Let us develop  $g p_n(d\alpha_g)$  in a Fourier series in  $P_K(d\alpha)$ . We have

$$(8) \quad g(x) p_n(d\alpha_g, x) = \frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} p_n(d\alpha, x) + d_{n+1} p_{n+1}(d\alpha, x) + \frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} p_{n+2}(d\alpha, x)$$

Unfortunately we cannot directly calculate  $d_{n+1}$ . Let us note that  $g(A \pm 1B) = 0$ . Hence

$$\frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} p_n(d\alpha, A \pm 1B) + d_{n+1} p_{n+1}(d\alpha, A \pm 1B) +$$

$$\frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} p_{n+2}(d\alpha, A \pm 1B) = 0.$$

-83-

Consequently

$$(9) \quad -d_{n+1} = \frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} \frac{p_n(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)} + \frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} \frac{p_{n+2}(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)}$$

and

$$\frac{Y_n^2(d\alpha)}{Y_n^2(d\alpha_g)} = \frac{Y_n(d\alpha)}{Y_{n+2}(d\alpha)} \left[ \frac{p_{n+2}(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)} - \frac{p_{n+2}(d\alpha, A-1B)}{p_{n+1}(d\alpha, A-1B)} \right] \cdot \left[ \frac{p_n(d\alpha, A-1B)}{p_{n+1}(d\alpha, A-1B)} - \frac{p_n(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)} \right]^{-1}$$

Letting  $n \rightarrow \infty$  and using Theorem 4.1.13 we obtain

$$\lim_{n \rightarrow \infty} \frac{Y_n^2(d\alpha)}{Y_n^2(d\alpha_g)} = \frac{1}{4} [\rho(A+1B) - \rho(A-1B)] \cdot [\rho^{-1}(A-1B) - \rho^{-1}(A+1B)]^{-1} = \frac{1}{4} \rho(A+1B) \rho(A-1B)$$

which proves the left side equality in (7). The right side equality in (7) follows from Lemma 4.2.2. Now we shall show that for every

$z \in \mathbb{C} \setminus \text{supp}(d\alpha) = \mathbb{C} \setminus \text{supp}(d\alpha_g)$  ( $z \neq A \pm 1B$ )

$$(10) \quad \lim_{n \rightarrow \infty} \frac{p_n(d\alpha_g, z)}{p_{n+1}(d\alpha_g, z)} = \rho(z)^{-1}.$$

If (10) holds then by Theorem 4.1.12  $\alpha_g \in M(0,1)$ . We obtain from (8) and (9)

-84-

$$\frac{g(z) p_n(d\alpha, z)}{p_n(d\alpha, z)} = \frac{Y_n(d\alpha)}{Y_{n+2}(d\alpha)} \left[ \frac{Y_{n+2}(d\alpha) Y_n(d\alpha)}{Y_n(d\alpha)} \right] \\
\left( \frac{Y_{n+2}(d\alpha) Y_n(d\alpha)}{Y_n(d\alpha)} \frac{p_n(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)} + \frac{p_{n+2}(d\alpha, A+1B)}{p_{n+1}(d\alpha, A+1B)} \right) \\
\cdot \frac{p_{n+1}(d\alpha, z)}{p_n(d\alpha, z)} + \frac{p_{n+2}(d\alpha, z)}{p_n(d\alpha, z)} \Bigg]$$

By (7) and Theorem 4.1.13 for  $z \in \mathbb{C} \setminus \text{supp}(d\alpha) \setminus \{A \pm iB\}$

$$(11) \quad \lim_{n \rightarrow \infty} \frac{g(z) p_n(d\alpha, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A+iB)\rho(A-iB)} \right|^{\frac{1}{2}}.$$

$$\cdot [\rho(z) - \rho(A+iB)][\rho(z) - \rho(A-iB)].$$

Now (10) follows from Theorem 4.1.13. Hence  $\alpha_g \in M(0, 1)$ .

Let us remark that by Theorem 4.1.13 (10) holds also for  $z = A \pm iB$ .

Lemma 13. Let  $\alpha \in M(0, 1)$ . Let  $g(x) = (x-A)(x-B)$  ( $A \neq B$ ) be positive on  $\text{supp}(d\alpha)$ . Then  $\alpha_g \in M(0, 1)$ .

$$\lim_{n \rightarrow \infty} \frac{Y_n(d\alpha_g)}{Y_n(d\alpha)} = 2 \left| \frac{1}{\rho(A)\rho(B)} \right|^{\frac{1}{2}} = \\
= \exp \left( -\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right),$$

further for every  $z \in \mathbb{C} \setminus (\text{supp}(d\alpha) \setminus \{A, B\})$

$$\lim_{n \rightarrow \infty} \frac{g(z) p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A)\rho(B)} \right|^{\frac{1}{2}}.$$

$$\cdot [\rho(z) - \rho(A)][\rho(z) - \rho(B)].$$

Proof. The proof of Lemma 12 can be repeated. Note that  $A, B \notin [-1, 1]$  but may belong to  $\Delta(d\alpha)$ .

Lemma 14. Let  $\alpha \in M(0, 1)$  and let  $g(x) = (x-A)^2$  where  $A \in \mathbb{R} \setminus \text{supp}(d\alpha)$ , that is  $A$  may belong to  $\Delta(d\alpha)$ . Then  $\alpha_g \in M(0, 1)$  and

$$(12) \quad \lim_{n \rightarrow \infty} \frac{Y_n(d\alpha_g)}{Y_n(d\alpha)} = 2 \left| \frac{1}{\rho(A)} \right| = \exp \left( -\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right).$$

Proof. The proof of Lemma 12 has to be modified. We have (8) and  $g(A) = g'(A) = 0$ . Hence

$$\frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} p_n(d\alpha, A) + d_{n+1} p_{n+1}(d\alpha, A) + \frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} p_{n+2}(d\alpha, A) = 0$$

and

$$\frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} p'_n(d\alpha, A) + d_{n+1} p'_{n+1}(d\alpha, A) + \frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} p'_{n+2}(d\alpha, A) = 0.$$

From here

$$(13) \quad -d_{n+1} = \frac{Y_n(d\alpha)}{Y_n(d\alpha_g)} \frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} + \frac{Y_n(d\alpha_g)}{Y_{n+2}(d\alpha)} \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)}$$

and

$$\frac{Y_n(d\alpha)}{Y_n(d\alpha)'} = \frac{Y_n(d\alpha)}{Y_{n+2}(d\alpha)} \left[ \frac{p'_{n+2}(d\alpha, A)}{p'_{n+1}(d\alpha, A)} - \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)} \right] \cdot \left[ \frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} - \frac{p'_n(d\alpha, A)}{p'_{n+1}(d\alpha, A)} \right]^{-1}.$$

Now (12) follows from Theorems 4.1.13, 4.1.16 and 4.1.17, further from Lemma 4.2.2. Using (8), (12), (13) and Theorem 4.1.13 we obtain by the same way as we did in the proof of Lemma 12 that

$$(14) \quad \lim_{n \rightarrow \infty} \frac{g(z) p_n(d\alpha, z)}{p_n(d\alpha, z)'} = \frac{1}{2} \left| \frac{1}{\rho(A)} \right| [\rho(z) - \rho(A)]^2$$

for  $z \in \mathbb{C} \setminus \text{supp}(d\alpha) \setminus \{A\}$  which together with Theorems 4.1.12 and 4.1.13 shows that  $\alpha_g \in M(0, 1)$ .

Theorem 15. Let  $\alpha \in M(0, 1)$ . Let

$$g(x) = A \prod_{k=1}^N (x - B_k)$$

be positive on  $\text{supp}(d\alpha)$ . Then  $\alpha_g \in M(0, 1)$ ,

$$\lim_{n \rightarrow \infty} \frac{Y_n(d\alpha)}{Y_n(d\alpha)'} = \exp \left( -\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right)$$

and for every  $z \in \mathbb{C} \setminus \text{supp}(d\alpha) \setminus \{B_k\}$

$$\lim_{n \rightarrow \infty} \frac{p_n(d\alpha, z)}{p_n(d\alpha, z)'} = \frac{1}{2} \exp \left( -\frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right).$$

$$\prod_{k=1}^N \frac{\rho(z) - \rho(B_k)}{z - B_k}.$$

-87-

Proof. Repeated application of Lemmas 10, 12, 13, 14, of Remark 11 and of formulas (11) and (14).

Definition 16. Let  $\beta \in S$ . Then the Szegő function  $D(d\beta, z)$  is defined by

$$D(d\beta, z) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \beta'(cost) \cdot \frac{1+ze^{-it}}{1-ze^{-it}} dt \right\}$$

for  $|z| < 1$ .

Properties 17.  $D \in H_2(|z| < 1)$ , for almost every  $t \in [-\pi, \pi]$

$$\lim_{r \rightarrow 1-0} D(d\beta, re^{it}) = D(d\beta, e^{it})$$

exists and  $|D(d\beta, e^{it})|^2 = \beta'(cost)$  for almost every  $t \in [-\pi, \pi]$ ,  $D(d\beta, z) \neq 0$  for  $|z| < 1$ ,  $D(d\beta, 0) > 0$ . (See e.g. Freud, Chapter V.)

Lemma 18. Let  $\beta \in S$ ,  $z \in \mathbb{C} \setminus [-1, 1]$ . Then

$$\lim_{n \rightarrow \infty} p_n(d\beta, z) \rho(z)^{-n} = \frac{1}{\sqrt{2\pi}} D(d\beta, \rho(z)^{-1})^{-1}$$

and the convergence is uniform for  $|\rho(z)| \geq R > 1$ .

Proof. See e.g. Freud.

Lemma 19. Let  $g(x) = A \prod_{k=1}^N (x - B_k)$  be positive on  $[-1, 1]$ . Then for  $z \in \mathbb{C} \setminus [-1, 1]$

$$(15) \quad D(g, \rho(z)^{-1}) = 2^N \exp \left( \frac{1}{2\pi} \int_{-1}^1 \log g(t) \frac{dt}{\sqrt{1-t^2}} \right).$$

$$\prod_{k=1}^N \frac{z - B_k}{\rho(z) - \rho(B_k)}.$$

Proof. Put in Theorem 15  $\alpha = \text{Tschubyshev weight}$  and use Lemma 18.

Then for  $z \in \mathbb{C} \setminus [-1, 1] \setminus \{B_k\}$  (15) holds and consequently it holds for every  $z \in \mathbb{C} \setminus [-1, 1]$ .

-88-

Let us note that - because of continuity arguments - (15) holds if  $g(\pm 1) = 0$ , further (15) holds for  $z = \infty$  if  $g$  is only nonnegative on  $[-1, 1]$ .

**Theorem 20.** Let  $\alpha \in M(0, 1)$  and let  $g$  be a polynomial which is positive on  $\text{supp}(d\alpha)$ . Then  $\sigma_g \in M(0, 1)$  and

$$\lim_{n \rightarrow \infty} \frac{Y_n(d\alpha)}{Y_n(d\sigma_g)} = D(g, 0),$$

$$\lim_{n \rightarrow \infty} \frac{P_n(d\alpha, z)}{P_n(d\sigma_g, z)} = D(g, \rho(z)^{-1})$$

for  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ .

**Proof.** By Theorem 15 and Lemma 19 the only thing which we have to show is that if  $g(B) = 0$  then

$$(16) \quad \lim_{n \rightarrow \infty} \frac{P_n(d\alpha, B)}{P_n(d\sigma_g, B)} = D(g, \rho(B)^{-1}).$$

Let  $\delta > 0$  be small enough. Then for  $|z - B| = \delta$

$$\lim_{n \rightarrow \infty} \frac{P_n(d\alpha, z)}{P_n(d\sigma_g, z)} = D(g, \rho(z)^{-1}),$$

that is by Theorem 4.1.13

$$\lim_{n \rightarrow \infty} \frac{P_{n-1}(d\alpha, z)}{P_n(d\sigma_g, z)} = \rho(z)^{-1} D(g, \rho(z)^{-1})$$

for  $|z - B| = \delta$ . Since  $P_{n-1}(d\alpha, z) = L_n(d\sigma_g, P_{n-1}(d\alpha, z))$  we have

$$\left| \frac{P_{n-1}(d\alpha, z)}{P_n(d\sigma_g, z)} \right| \leq \frac{Y_{n-1}(d\alpha)}{Y_n(d\sigma_g)} \frac{1}{\varepsilon} \sum_{k=1}^n \lambda_{kn}^{(d\alpha)} |P_{n-1}(d\sigma_g, x_{kn})|.$$

$$|P_{n-1}(d\alpha, x_{kn})| \leq \frac{Y_{n-1}(d\alpha)}{Y_n(d\sigma_g)} \frac{1}{\varepsilon} \left( \int_{-\infty}^{\infty} P_{n-1}^2(d\alpha, t) d\sigma_g(t) \right)^{\frac{1}{2}}$$

$$\leq \frac{Y_{n-1}(d\alpha)}{Y_n(d\sigma_g)} \frac{1}{\varepsilon} \max_{t \in \text{supp}(d\alpha)} |g(t)|^{\frac{1}{2}}$$

for  $n \geq N$  where  $\varepsilon$  and  $N$  are defined by Theorem 3.3.8. Since both

$P_{n-1}(d\alpha, z)/P_n(d\sigma_g, z)$  and  $\rho(z)^{-1} D(g, \rho(z)^{-1})$  are analytic in  $|z - B| \leq \delta$

if  $\delta \leq \varepsilon$  and  $n \geq N$  we can apply Cauchy's integral formula and Lebesgue's theorem about  $\lim_{n \rightarrow \infty} \int_n f = \int \lim_{n \rightarrow \infty} f_n$  and we obtain

$$\lim_{n \rightarrow \infty} \frac{P_{n-1}(d\alpha, B)}{P_n(d\sigma_g, B)} = \rho(B)^{-1} D(g, \rho(B)^{-1}).$$

Thus by Theorem 4.1.13 (16) holds.

Now we can easily generalize Theorem 8.

**Theorem 21.** Let  $\alpha \in M(0, 1)$ . Let  $g$  be a polynomial which is positive on  $\text{supp}(d\alpha)$ . Then for every  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \frac{\lambda_n^{(d\alpha, z)}}{\lambda_n^{(d\sigma_g, z)}} = D(g, \rho(z)^{-1})^2$$

and

$$(17) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n^{(d\alpha, z)}}{\lambda_n^{(d\sigma_g, z)}} = |D(g, \rho(z)^{-1})|^2.$$

**Proof.** Apply Theorems 20 and 4.1.11.



Remark 22. Let us put  $\rho(z)^{-1} = re^{i\theta}$  ( $0 < r < 1$ ) in (17). Then  $z = \frac{1}{2}(re^{i\theta} + r^{-1}e^{-i\theta}) \xrightarrow{r \rightarrow 1-0} \cos \theta$ . By Properties 17 for almost every  $\theta \in [-\pi, \pi]$

$$\lim_{r \rightarrow 1-0} \lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, z)}{\lambda_n(dx, z)} = g(x) \quad (x = \cos \theta)$$

which suggests

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, x)}{\lambda_n(dx, x)} = g(x) \quad (-1 \leq x \leq 1).$$

Property 23. Let  $z = re^{i\theta}$   $0 \leq r < 1$ . Then

$$|D(d\beta, z)|^2 = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \beta'(cost) \frac{1-r^2}{1-2r \cos(\theta-t)+r^2} dt \right\}.$$

(See e.g. Freud, Chapter V.)

Lemma 24. Let  $f$  be Riemann integrable on  $\Delta \supset [-1, 1]$  and let  $f(x) \geq C > 0$  for  $x \in \Delta$ . Let  $0 < R < 1$  be fixed. Then for every  $\epsilon > 0$  there exist two polynomials  $\pi_1$  and  $\pi_2$  such that

$$\frac{C}{2} \leq \pi_1(x) \leq f(x) \leq \pi_2(x)$$

for  $x \in \Delta$  and

$$|D(\pi_2, z)|^2 (1-\epsilon) \leq |D(t, z)|^2 \leq |D(\pi_1, z)|^2 (1+\epsilon)$$

for  $|z| < R$ .

Proof. Let  $\epsilon > 0$ . We construct a polynomial  $\pi_2$  such that

$$f(x) \leq \pi_2(x) \quad (x \in \Delta)$$

-91-

and

$$\int_{-\pi}^{\pi} |\pi_2(cost) - f(cost)| dt < \epsilon.$$

(See Szegő, I. 5.) Then by Property 23

$$|D(\pi_2, z)|^2 = |D(\pi_2 f^{-1}, z)|^2 |D(f, z)|^2$$

and by Jensen's inequality

$$|D(\pi_2 f^{-1}, z)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(cost)}{f(cost)} \frac{1-r^2}{1-2r \cos(\theta-t)+r^2} dt$$

( $z = re^{i\theta}$ ). Hence

$$|D(\pi_2 f^{-1}, z)|^2 \leq 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(cost) - f(cost)}{f(cost)} dt.$$

$$\frac{1-r^2}{1-2r \cos(\theta-t)+r^2} dt \leq 1 + \text{const} \cdot \epsilon$$

for  $|z| \leq R$ . The second part of the lemma can be proved in the same way. Since  $f(x) \geq C > 0$  for  $x \in \Delta$  we can choose  $\pi_1$  so that  $\pi_1(x) \geq \frac{C}{2}$  for  $x \in \Delta$ . (See Szegő, I. 5.)

Theorem 25. Let  $\alpha \in M(0, 1)$ . Let  $g(\geq 0)$  be  $d\alpha$  measurable,  $g \neq 1$  be bounded on  $\text{supp}(d\alpha)$  and  $g$  be Riemann integrable on  $[-1, 1]$ . Then for every  $z \in \mathbb{D} \setminus \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, z)}{\lambda_n(dx, z)} = |D(g, \rho(z)^{-1})|^2.$$

-92-

Proof. By the assumptions  $\omega_g$  is a weight, since  $g^{-1} \in L^1 d\alpha$ . Recall that

$$\lambda_n(d\alpha, z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |(1+(z-t)\pi_{n-2}(t))|^2 d\alpha(t).$$

Then from  $d\alpha \leq d\beta$  it follows that  $\lambda_n(d\alpha, z) \leq \lambda_n(d\beta, z)$  for every  $z \in \mathbb{C}$ . Let us construct two functions  $f_1$  and  $f_2$  such that both  $f_1$  and  $f_2$  are Riemann integrable on  $\Delta(d\alpha)$ ,  $f_1(x) = f_2(x) = g(x)$  for  $x \in [-1, 1]$  and

$$0 < c_1 \leq f_1(x) \leq g(x) \leq f_2(x) \leq c_2 < \infty$$

for  $x \in \Delta(d\alpha)$ . We can do this by Theorem 3.3.7. Let  $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$ .

Then  $|\rho(z)^{-1}| < 1$ . Let  $\varepsilon > 0$ . Then by Lemma 24 we can find two polynomials  $\pi_1$  and  $\pi_2$  such that  $\pi_1(x) \geq c_1/2$ ,  $\pi_2(x) \geq c_1/2$  for  $x \in \Delta(d\alpha)$ ,

$$\lambda_n(d\alpha, z)^{-1} \leq \lambda_n(d\alpha, z) \leq \lambda_n(d\alpha, z)$$

and

$$(1-\varepsilon)|D(\pi_2, \rho(z)^{-1})|^2 \leq |D(g, \rho(z)^{-1})|^2 \leq (1+\varepsilon)|D(\pi_1, \rho(z)^{-1})|^2.$$

Thus by Theorem 21

$$|D(g, \rho(z)^{-1})|^2 \frac{1}{1+\varepsilon} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, z)}{\lambda_n(d\alpha, z)} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, z)}{\lambda_n(d\alpha, z)} \leq \frac{1}{1-\varepsilon} |D(g, \rho(z)^{-1})|^2.$$

Now let  $\varepsilon \rightarrow 0$ .

Theorem 26. Let  $\alpha \in M(0, 1)$  and  $g$  be as in Theorem 25. Then

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha)}{\lambda_n(d\alpha_g)} = D(g, 0),$$

and so

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}(d\alpha_g)}{\lambda_n(d\alpha_g)} = \frac{1}{2}.$$

Proof. Recall that from  $d\alpha \leq d\beta$  follows  $\lambda_n(d\beta) \leq \lambda_n(d\alpha)$ . Now we can repeat the proof of Theorem 25, the only difference is that this time we apply Theorem 20 instead of Theorem 21.

Theorem 27. Let  $\alpha \in M(0, 1)$  and  $g$  be as in Theorem 25. Then  $\alpha_g \in M(0, 1)$

Proof. If we could directly calculate  $\alpha_n(d\alpha_g)$  the proof would probably be nice. Unfortunately we cannot do this. Let  $x \notin \text{supp}(d\alpha)$ . Then by Theorem 25

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_{n+1}(d\alpha_g, x)} \cdot \frac{\lambda_{n+1}(d\alpha, x)}{\lambda_n(d\alpha, x)} = 1,$$

that is

$$\lim_{n \rightarrow \infty} \frac{1 + \frac{p_n^2(d\alpha_g, x)}{\lambda_n(d\alpha_g, x)}}{2} \cdot \frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha, x)} = 1.$$

Thus by Theorem 4.1.11

$$\lim_{n \rightarrow \infty} \frac{p_n^2(d\alpha_g, x)}{\lambda_n(d\alpha_g, x)} \lambda_n(d\alpha_g, x) = \rho(x)^2 - 1.$$

Using Theorem 26 we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} = \frac{1}{p(x)^2 - 1} = -2 \frac{d}{dx} p(x)^{-1}$$

for  $x \notin \text{supp}(d\alpha)$ . By Lebesgue's dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{x - x_{kn}} = \frac{2}{p(x)} = \frac{2}{\pi} \int_{-1}^1 \frac{\sqrt{1-t^2}}{x-t} dt$$

for  $x \notin \Delta(d\alpha)$ . If  $f$  is continuous on  $\Delta(d\alpha)$  then for every  $\varepsilon > 0$  we can find a function  $F$  of the form

$$F(t) = \sum_{j=1}^N a_j \frac{1}{x_j - t}$$

where  $a_j \in \mathbb{C}$  and  $x_j \in \mathbb{R} \setminus \Delta(d\alpha)$  such that

$$\max_{t \in \Delta(d\alpha)} |F(t) - f(t)| \leq \varepsilon.$$

(See e.g. Achieser, section of problems). Hence if  $f$  is continuous on  $\Delta(d\alpha)$  then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn} \frac{f(x_{kn})}{x_{kn}} \frac{p_{n-1}^2(d\alpha, x_{kn})}{x_{kn}} = \frac{2}{\pi} \int_{-1}^1 f(t) \sqrt{1-t^2} dt.$$

Consequently by Theorem 3.2.4  $\alpha_g \in M(0, 1)$ .

Remark 2.8. Later we shall show (with the aid of the Pollaczek polynomials), that if  $w$  is defined by

$$w(x) = \exp(-(1-x^2)^{-\frac{1}{2}})$$

for  $-1 \leq x \leq 1$  and  $\text{supp}(w) = [-1, 1]$  then  $w \in M(0, 1)$ . Consequently by

the previous theorem  $gw \in M(0, 1)$  if  $g > 0$  is continuous on  $[-1, 1]$ . Let us remark that the above  $w$  is the "nicest" weight which does not belong to  $S$ .

Theorem 2.9. Let  $\alpha \in M(0, 1)$  and let  $g$  satisfy the conditions of Theorem 2. Let  $K \subset \mathbb{C} \cup \{\infty\} \setminus \text{supp}(d\alpha)$  be an arbitrary closed set. Then

$$(18) \quad \lim_{n \rightarrow \infty} \frac{p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = D(g, \rho(z))^{-1-1}$$

uniformly for  $z \in K$ .

Proof. If  $z \in \mathbb{R} \setminus \text{supp}(d\alpha)$  then (18) follows immediately from Theorem 2.5, 2.7, 3.3.8 and 4.1.11. Let  $K^*$  be a region in  $\mathbb{C} \cup \{\infty\}$  such that  $K \subset K^*$ ,  $K^* \cap \text{supp}(d\alpha) = \emptyset$  and  $K^* \cap \mathbb{R} \neq \emptyset$ . By Theorem 3.3.8 the functions  $p_n(d\alpha_g, z) p_n(d\alpha, z)^{-1}$  are analytic in  $K^*$ . If we can show that

$$(19) \quad \frac{|p_n(d\alpha_g, z)|}{|p_n(d\alpha, z)|} \leq \text{const}$$

for  $z \in K^*$  and  $n = N, N+1, \dots$  where  $N = N(K^*)$  then the theorem will follow from Vitali's theorem. Let  $d_N$  be defined by  $d_N = \text{dist}(K^*, \{x_{kn}(d\alpha)\}_{n=N, k=1}^\infty\}$ . By Theorem 3.3.8  $d_N > 0$  for some  $N \in \mathbb{N}$ . Let  $n \geq N$  and  $z \in K^*$ . Then

$$\begin{aligned} |p_n(d\alpha_g, z)|^2 &\leq \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha, t) d\alpha(t) \leq \\ &\leq C \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha_g, t) d\alpha_g(t) \end{aligned}$$

where  $C^{-1} = \inf_{t \in \text{supp}(d\alpha)} g(t)$ . Hence

$$|p_n(d\sigma, z)|^2 \leq C |p_n(d\sigma, z)|^2 + C \lambda_n(d\sigma, z)^{-1}$$

Further we have

$$\begin{aligned} \lambda_n(d\sigma, z)^{-1} &= \sum_{k=1}^n \frac{|f_{kn}(d\sigma, z)|^2}{\lambda_{kn}(d\sigma)} < \\ &\leq \frac{Y_{n-1}(d\sigma)^2}{Y_n(d\sigma)^2} |p_n(d\sigma, z)|^2 d_N^{-2} \end{aligned}$$

Consequently (19) is satisfied with  $\text{const} = [C(1 + d_N^{-2} |\Delta(d\sigma)|^2 \cdot 0.25)]^{\frac{1}{2}}$ .

## 6.2. A Sequence of Positive Operators

Using the well known formula

$$f_{kn}(d\sigma, x) = \lambda_{kn}(d\sigma) K_n(d\sigma, x, x_{kn})$$

we obtain

$$F_n(d\sigma, f, x) = \lambda_n(d\sigma, x) \sum_{k=1}^n \lambda_{kn}(d\sigma) f(x_{kn}) K_n^2(d\sigma, x, x_{kn})$$

which is the Riemann-Stieltjes sum for

$$G_n(d\sigma, f, x) = \lambda_n(d\sigma, x) \int_{-\infty}^{\infty} f(t) K_n^2(d\sigma, x, t) d\sigma(t).$$

For  $z \in \mathbb{C}$  we put

$$G_n(d\sigma, f, z) = \lambda_n^*(d\sigma, z) \int_{-\infty}^{\infty} f(t) K_n^2(d\sigma, z, t) d\sigma(t).$$

(See 4. 1).

Properties 1. (i) If  $f(x) \equiv 1$  then  $G_n(f, x) \equiv 1$ . (ii) If  $f(x) \geq 0$  for  $x \in \text{supp}(d\sigma)$  then  $G_n(f, x) \geq 0$  for  $x \in \mathbb{R}$ . (iii)  $G_n$  is a rational function of degree  $(2n-2, 2n-2)$  where the denominator does not depend on  $f$ .

Theorem 2. Let  $\sigma \in M(0, 1)$ . Let  $f$  be  $d\sigma$  measurable and bounded on  $\text{supp}(d\sigma)$ . Then for each  $x \in \text{supp}(d\sigma) \setminus [-1, 1]$

$$(1) \quad \lim_{n \rightarrow \infty} G_n(d\sigma, f, x) = f(x).$$

If  $x \in [-1, 1]$  and  $f$  is continuous at  $x$  then (1) holds. If  $f$  is continuous on  $\Delta \subset (-1, 1)$  then (1) holds uniformly for  $x \in \Delta$ . If  $f$  is continuous on  $\text{supp}(d\sigma)$  and  $z \in \mathbb{C} \setminus \text{supp}(d\sigma)$  then



$$(2) \quad \lim_{n \rightarrow \infty} G_n(d\sigma, f, z) = \frac{\sqrt{z^2 - 1}}{\pi} \int_{-1}^1 \frac{f(t) \sqrt{1-t^2}}{(z-t)\sqrt{1-t^2}} dt.$$

Here  $\sqrt{z^2 - 1} > 0$  for  $z > 1$ .

Proof. (i) Let  $x \in \text{supp}(d\sigma) \setminus [-1, 1]$ . Then by Theorem 3.3.7  $x$  is an isolated point of  $\text{supp}(d\sigma)$ . Hence there exists  $\epsilon > 0$  such that

$$G_n(f, x) = f(x) \frac{\sigma(x+0) - \sigma(x-0)}{\lambda_n(x)} + \lambda_n(x).$$

$$\cdot \int_{|x-t| \geq \epsilon} \frac{f(t) K_n^2(x, t)}{\lambda_n(x)} d\sigma(t).$$

Here the first term converges to  $f(x)$  when  $n \rightarrow \infty$ . (See Freud, §II.2,

$\text{supp}(d\sigma)$  is compact!) Remembering that

$$K_n(x, t) = \frac{y_{n-1}}{y_n} \frac{p_{n-1}(t) p_n(x) - p_n(t) p_{n-1}(x)}{x-t}$$

and using Theorem 4.1.11 we see that

$$(3) \quad \lim_{n \rightarrow \infty} \lambda_n(x) \int_{|x-t| > \epsilon} \frac{f(t) K_n^2(x, t)}{\lambda_n(x)} d\sigma(t) = 0.$$

(ii) Let  $x \in [-1, 1]$ . Then for every  $\epsilon > 0$  (3) is satisfied and the convergence is uniform for  $x \in \Delta \subset (-1, 1)$ . Thus by Properties 1 the usual machinery of positive operators can be applied. We do not go into details.

(iii) Let  $z \in \mathbb{C} \setminus \text{supp}(d\sigma)$ . By Titchmarsh's theorem we can suppose that  $f$  is continuous on  $\Delta(d\sigma)$ . The function  $(z-t)^{-2}$  restricted to  $\text{supp}(d\sigma)$  is continuous and we can extend it to a function  $g$  which is continuous on

$\Delta(d\sigma)$ . We have

-99-

$$G_n(f, z) = \lambda_n(z) \frac{y_{n-1}}{y_n} \int_{-\infty}^{\infty} \frac{f(t) g(t) [p_{n-1}(t) p_n(z) -$$

$$- p_n(t) p_{n-1}(z)]^2 d\sigma(t) =$$

$$= \frac{y_{n-1}}{y_n} \left[ \lambda_n(z) p_n^2(z) \int_{-\infty}^{\infty} f(t) g(t) p_{n-1}^2(t) d\sigma(t) + \right.$$

$$+ \lambda_n^*(z) p_n^2(z) \frac{p_{n-1}^2(z)}{p_n(z)} \int_{-\infty}^{\infty} f(t) g(t) p_n^2(t) d\sigma(t) -$$

$$- 2\lambda_n^*(z) p_n^2(z) \frac{p_{n-1}(z)}{p_n(z)} \int_{-\infty}^{\infty} f(t) g(t) p_{n-1}(t) p_n(t) d\sigma(t) \Big].$$

Now we apply Theorems 4.1.11, 4.1.13 and 4.2.13. We obtain

$$\lim_{n \rightarrow \infty} G_n(f, z) = \frac{1}{4\pi} [\rho^2(z) - 1] [1 + \rho^{-2}(z)] \int_{-1}^1 \frac{f(t)}{(z-t)^2 \sqrt{1-t^2}} dt -$$

$$- \frac{1}{2\pi} [\rho^2(z) - 1] \rho^{-1}(z) \int_{-1}^1 \frac{tf(t)}{(z-t)^2 \sqrt{1-t^2}} dt.$$

$$\text{But } [\rho^2(z) - 1] [1 + \rho^{-2}(z)] = 4z\sqrt{z^2 - 1} \text{ and } [\rho^2(z) - 1] \rho^{-1}(z) = 2\sqrt{z^2 - 1}.$$

Hence

$$\lim_{n \rightarrow \infty} G_n(f, z) = \frac{\sqrt{z^2 - 1}}{\pi} \int_{-1}^1 \frac{f(t)}{(z-t)^2 \sqrt{1-t^2}} dt.$$

Let us note that once (2) holds for continuous functions then it also holds for Riemann integrable functions if  $x \in \mathbb{R} \setminus \text{supp}(d\sigma)$ . We shall not go into details since in the following we shall concentrate on convergence of  $G_n(d\sigma, f, x)$  for  $x \in \text{supp}(d\sigma)$ . The following theorem explains why we

introduced the operators  $G_n(d\alpha, t)$  and why we should investigate them for as many weights  $\alpha$  as possible.

**Theorem 3.** Let  $g(\geq 0) \in L^1_{d\alpha}$ . If  $\alpha_g$  is a weight then

$$(4) \quad \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \leq G_n(d\alpha, g, x)$$

for  $x \in \mathbb{R}$  and if  $g^{-1} \in L^1_{d\alpha}$  then

$$(5) \quad G_n^{-1}(d\alpha, g^{-1}, x) \leq \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)}$$

for  $x \in \mathbb{R}$ .

Before the proof let us remark that if  $\text{supp}(d\alpha)$  is compact and  $g^{-1} \in L^1_{d\alpha}$  then  $\alpha_g$  is a weight.

Proof. From

$$\lambda_n(d\alpha, x) = \min_{\pi_{n-1}} \pi_{n-1}^{-2}(x) \int_{-\infty}^{\infty} \pi_{n-1}^2(t) d\alpha(t)$$

follows that

$$\begin{aligned} \lambda_n(d\alpha_g, x) &\leq K_n^{-2}(d\alpha, x, x) \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t) d\alpha(t) = \\ &= \lambda_n^2(d\alpha, x) \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t) g(t) d\alpha(t) = \lambda_n(d\alpha, x) G_n(d\alpha, g, x). \end{aligned}$$

On the other hand

$$\pi_{n-1}^{-1}(x) = \int_{-\infty}^{\infty} K_n(d\alpha, x, t) \pi_{n-1}^{-1}(t) d\alpha(t).$$

Thus

$$\pi_{n-1}^2(x) \leq \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t) g^{-1}(t) d\alpha(t).$$

$$\int_{-\infty}^{\infty} \pi_{n-1}^2(t) g(t) d\alpha(t) = \lambda_n^{-1}(d\alpha, x) G_n(d\alpha, g^{-1}, x).$$

$$\int_{-\infty}^{\infty} \pi_{n-1}^2(t) d\alpha_g(t),$$

that is

$$\lambda_n^{-1}(d\alpha, g, x) \leq \lambda_n^{-1}(d\alpha, x) G_n(d\alpha, g^{-1}, x).$$

From Theorems 2 and 3 we could immediately obtain limit relations

for

$$\frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)}$$

when both  $g$  and  $g^{-1}$  are bounded on  $\text{supp}(d\alpha)$ . This condition however may be weakened by using the following two results.

**Lemma 4.** Let  $\alpha \in M(0, 1)$ . Let  $\{k_n\}$  be a sequence of natural integers which is bounded:  $k_n \leq k$  for every  $n$ . Then

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, x)}{\lambda_{n+k_n}(d\alpha, x)} = 1$$

for every  $x \in \text{supp}(d\alpha)$  and the convergence is uniform for  $x \in \Delta \subset (-1, 1)$ .

Proof. Since

$$1 \leq \frac{\lambda_n(x)}{\lambda_{n+k_n}(x)} \leq \frac{\lambda_n(x)}{\lambda_{n+k}(x)} = \prod_{j=n}^{n+k-1} \frac{\lambda_j(x)}{\lambda_{j+1}(x)}$$

we have to consider only  $\lambda_n(x)/\lambda_{n+1}(x)$  which equals

$$1 + \lambda_n(d\alpha, x) \frac{P_2^2(d\alpha, x)}{P_1^2(d\alpha, x)}.$$

Now we apply Theorem 4.1.11.

Theorem 5. Let  $g(\geq 0) \in L_{d\alpha}^1$ . Let  $\alpha_g$  be a weight. Then for every polynomial  $P_1$  of degree  $m_1$

$$(6) \quad \frac{P_1^2(x) \lambda_n(d\alpha_g, x)}{\lambda_{n-m_1}(d\alpha, x)} \leq G_{n-m_1}(d\alpha, gP_1^2, x) \quad (n > m_1).$$

If  $P_2$  is a polynomial of degree  $m_2$  such that  $P_2^2 g^{-1} \in L_{d\alpha}^1$ , then

$$(7) \quad P_2^2(x) G_{n+m_2}^{-1}(d\alpha, g^{-1} P_2^2, x) \leq \frac{\lambda_n(d\alpha_g, x)}{\lambda_{n+m_2}(d\alpha, x)}.$$

Let us note that if  $\text{supp}(d\alpha)$  is compact and  $P_2^2 g^{-1} \in L_{d\alpha}^1$  for some polynomial  $P_2$  then  $\alpha_g$  is a weight.

Proof. (6) follows from

$$\lambda_n(d\alpha_g, x) \leq P_1^{-2}(x) K_{n-m_1}^{-2}(d\alpha, x, x) \int_{-\infty}^{\infty} P_1^2(t) K_{n-m_1}^2(d\alpha, x, t) \cdot d\alpha(t)$$

whenever  $n > m_1$ . Further for every  $\pi_{n-1}$

$$\pi_{n-1}(x) P_2(x) = \int_{-\infty}^{\infty} \pi_{n-1}(t) P_2(t) K_{n+m_2}(d\alpha, x, t) d\alpha(t),$$

that is

$$\pi_{n-1}^2(x) P_2^2(x) \leq \int_{-\infty}^{\infty} \pi_{n-1}^2(t) g(t) d\alpha(t) \cdot \int_{-\infty}^{\infty} P_2^2(t) g^{-1}(t) K_{n+m_2}^2(d\alpha, x, t) d\alpha(t)$$

which implies (7).

-103-

Theorem 6. Let  $\alpha \in M(0, 1)$ . Let  $g(\geq 0) \in L_{d\alpha}^1$  and suppose that there exist two polynomials  $P_1$  and  $P_2$  such that  $gP_1^2$  and  $g^{-1}P_2^2$  are bounded on  $\text{supp}(d\alpha)$ . Then

(i) for every  $x \in \text{supp}(d\alpha) \setminus [-1, 1]$

$$(8) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} = g(x).$$

(ii) if  $x \in [-1, 1]$  and  $g$  is continuous at  $x$  then (8) holds.

(iii) if  $g$  is continuous on  $\Delta \subset (-1, 1)$  and  $g(t) > 0$  for  $t \in \Delta$  then (8) is satisfied uniformly for  $x \in \Delta$ .

Proof. Let first  $x \in \text{supp}(d\alpha) \setminus [-1, 1]$ . Then by Theorem 3.3.7  $x$  is an isolated point of  $\text{supp}(d\alpha)$ . Hence  $g$  must be finite at  $x$  and then we can suppose that  $P_1$  does not vanish at  $x$ . We obtain from Theorems 2, 5 and Lemma 4 that

$$(9) \quad \limsup_{n \rightarrow \infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \leq g(x)$$

which implies (8) if  $g(x) = 0$ . If  $g(x) > 0$  then we can assume that  $P_2$  does not vanish at  $x$ . Then by the same argument

$$(10) \quad \liminf_{n \rightarrow \infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \geq g(x).$$

Let now  $x \in [-1, 1]$ . If  $g$  is continuous at  $x$  then  $g(x) < \infty$  and thus we can suppose that  $P_1(x) > 0$ . Hence (9) holds again. If  $g(x) = 0$  then (8) follows from (9). If  $g(x) > 0$  then we can suppose  $P_2(x) > 0$  which implies (10). If  $g$  is continuous on  $\Delta \subset (-1, 1)$  then the above argument can be used if only  $g$  is positive on  $\Delta$ .

-104-

In order to illustrate the strength of this theorem we give a few examples.

Definition 7.  $u$  denotes the Jacobi weight, that is  $\text{supp}(u) = [-1, 1]$  and

$$u(x) = u^{(a,b)}(x) = (1-x)^a (1+x)^b$$

for  $-1 \leq x \leq 1$  where  $a, b > -1$ . Hence  $u^{(-\frac{1}{2}, -\frac{1}{2})} = v$ .

In the following it will always be clear if  $a, b$  are related with  $u^{(a,b)}$  or  $M(a, b)$ .

Example 8. Let  $\alpha$  be the Tschebyshev weight ( $d\alpha(x) = v(x)dx$ ) and let  $w(x)\sqrt{1-x^2} = g(x)$  be positive and continuous on  $[-1, 1]$ . Then  $w = v \cdot g$ . Further, by easy calculation,

$$\lambda_n^{(-\frac{1}{2}, -\frac{1}{2})}(v, x) = \frac{1}{\pi} \left[ n - \frac{1}{2} + \frac{1}{2} U_{2n-2}(x) \right]$$

where  $U_n$  is the Tschebyshev polynomial of second kind. Hence for every  $x \in [-1, 1]$

$$\lim_{n \rightarrow \infty} \left[ n - \frac{1}{2} + \frac{1}{2} U_{2n-2}(x) \right] \lambda_n(w, x) = \pi w(x) \sqrt{1-x^2} = \pi g(x)$$

(later we shall show that the convergence is uniform for  $x \in [-1, 1]$ ), in particular

$$\lim_{n \rightarrow \infty} n \lambda_n(w, \pm 1) = \frac{\pi}{2} g(\pm 1).$$

Example 9. Let  $w = \varphi u$  where  $\varphi > 0$  is continuous on  $[-1, 1]$ . Then

$$\lim_{n \rightarrow \infty} n \lambda_n(w, x) = \pi \sqrt{1-x^2} w(x)$$

uniformly for  $x \in \Delta \subset (-1, 1)$ . This is, of course, not new.

Example 10. Let  $b > 0$ ,  $\text{supp}(w) = [-b, b]$  and  $w > 0$  be continuous on  $[-b, b]$ . Then

$$w^{(b)}(x) = w(bx)$$

is a weight on  $[-1, 1]$ . From the definition of Christoffel function we obtain

$$\lambda_n(w, x) = b \lambda_n(w^{(b)}, xb^{-1}).$$

Hence

$$\lim_{n \rightarrow \infty} n \lambda_n(w, x) = \pi \sqrt{b^2 - x^2} w(x)$$

uniformly for  $x \in \Delta \subset (-b, b)$ .

Example 11. Let  $w$  be continuous on  $[-1, 1]$  and  $w(x) > 0$  for  $x \in (-1, 1)$ . Let  $\epsilon > 0$ ,  $\delta > 0$  and  $\Delta = \{[-1+\delta, 1-\delta]\}$ . Then

$$1_{\Delta}(x) w(x) \leq w(x) \leq w(x) + \epsilon$$

for  $-1 \leq x \leq 1$ . Hence

$$n \lambda_n(1_{\Delta} w, x) \leq n \lambda_n(w, x) \leq n \lambda_n(w + \epsilon, x),$$

where  $\text{supp}(w + \epsilon) = \text{supp}(w)$  is assumed. Thus by the previous examples we have

$$\liminf_{n \rightarrow \infty} n \lambda_n(1_{\Delta} w, x) \leq \limsup_{n \rightarrow \infty} n \lambda_n(w, x) \leq \pi \sqrt{1-x^2} [w(x) + \epsilon]$$

uniformly for  $x \in \Delta_1 \subset \Delta^0$ . Since  $\epsilon > 0$  and  $\delta > 0$  are arbitrary we obtain



$$\lim_{n \rightarrow \infty} \lambda_n(w, x) = \pi \sqrt{1-x^2} w(x)$$

uniformly for  $x \in \Delta_1 \subset (-1, 1)$ . Recall that  $w(\pm 1)$  may vanish.

**Definition 12.** Let  $a, b \in \mathbb{R}$  with  $a > |b|$ . The Pollaczek weight  $w^{(a, b)}$  is defined by  $\text{supp}(w^{(a, b)}) = [-1, 1]$  and

$$w^{(a, b)}(\cos \theta) =$$

$$= 2 \exp \left\{ \frac{\theta}{\sin \theta} (a \cos \theta + b) \right\} [1 + \exp \left\{ \frac{\pi}{\sin \theta} (a \cos \theta + b) \right\}]^{-1},$$

$\theta \in [0, \pi]$ ,  $x = \cos \theta$ .

### Properties 13.

(i) We have

$$\alpha_n(w^{(a, b)}) = -\frac{b}{2n+1+a} = -\frac{b}{2n} + O\left(\frac{1}{n}\right)$$

for  $n = 0, 1, 2, \dots$  and

$$\frac{\gamma_{n-1}(w^{(a, b)})}{\gamma_n(w^{(a, b)})} = \frac{n}{2\sqrt{(n+\frac{a}{2})^2 - 4}} = \frac{1}{2} \cdot \frac{1}{4an} + O\left(\frac{1}{n}\right)$$

for  $n = 1, 2, \dots$  (See Szegő, Appendix\*)

\* Let us note that the formula (1.7) in the Appendix of Szegő's book is not quite correct, it should be written as

$$\begin{aligned} n P_n(x; a, b) &= \\ &= [(2n-1+a)x + b] P_{n-1}(x; a, b) - \\ &- (n-1) P_{n-2}(x; a, b), \quad n = 2, 3, 4, \dots \end{aligned}$$

(ii)  $w^{(a, b)} \in M(0, 1)$  but  $w^{(a, b)} \notin S$ .

(iii)  $(p_n^2(w^{(a, b)}, x))$  is uniformly bounded for  $x \in \Delta \subset (-1, 1)$ . To prove this use Theorem 3.1.11 and Example 11.

(iv) Let

$$\varphi(x) = w^{(a, b)}(x) \exp \left\{ \frac{(a+b)\pi}{\sqrt{2}\sqrt{1-x^2}} + \frac{(a-b)\pi}{\sqrt{2}\sqrt{1+x}} \right\}.$$

Then  $\varphi$  is continuous and positive on  $[-1, 1]$ .

(v) Let  $w^{(a)} = w^{(a, 0)}$ . Then  $w^{(a)}$  is even and

$$\lim_{n \rightarrow \infty} \frac{p_n^2(w^{(a)}, 1)}{\sqrt{n} \exp[4\sqrt{a}\sqrt{n}]} = \frac{1}{4\pi} e^{-a} \frac{1}{\sqrt{a}}.$$

(See Szegő, Appendix.)

(vi)

$$\lim_{n \rightarrow \infty} \frac{\gamma_n(w^{(a, b)})}{2^n n^{\frac{a}{2}}} = \Gamma\left(\frac{a+1}{2}\right)^{-1}$$

where  $\Gamma$  denotes the  $\Gamma$  function of Euler. (See Szegő, Appendix.)

(vii)

$$\lim_{n \rightarrow \infty} \lambda_n(w^{(a)}, x)n = \pi \sqrt{1-x^2} w^{(a)}(x)$$

uniformly for  $x \in \Delta \subset (-1, 1)$  and

$$\lim_{n \rightarrow \infty} \lambda_n(w^{(a)}, \pm 1)n \exp \{4\sqrt{a}\sqrt{n}\} = 2\pi e^a.$$

The first limit relation follows from Example 11, the second one from (v) by a standard calculation.

(viii) Let  $z \in \mathbb{C} \setminus [-1, 1]$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n(w(a, b), z) \rho^{-n}(z) n^{\frac{az+b}{2\sqrt{z^2-1}}} &= \\ &= \Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right)^{-1} \left[ \frac{\sqrt{z^2-1}}{\rho(z)} \right]^{-\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}} \end{aligned}$$

(See Szegő, Appendix.)

(ix) If  $z \in \mathbb{C} \setminus [-1, 1]$  then

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n^*(w(a, b), z)^{-1} \rho^{-2n}(z) n^{\frac{az+b}{2\sqrt{z^2-1}}} &= \\ &= \Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right)^{-2} \frac{1}{4(z^2-1)} \left[ \frac{\sqrt{z^2-1}}{\rho(z)} \right]^{\frac{az+b}{2\sqrt{z^2-1}}} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n(w(a, b), z)^{-1} |\rho(z)|^{-2n} n^{\frac{az+b}{2\sqrt{z^2-1}}} &= \\ &= \frac{1}{|\rho(z)|^2-1} \left| \Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right) \right|^{-2} \left[ \frac{\sqrt{z^2-1}}{\rho(z)} \right]^{-\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}} \end{aligned}$$

These follow from (ii), (viii) and Theorem 4.1.11.

Example 14. Let  $w$  be defined by  $\text{supp}(w) = [-1, 1]$  and

$$(11) \quad w(x) = \exp\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

-109-

for  $-1 \leq x \leq 1$ . By Property 13(iv)

$$g(x) = w(x) w\left(\frac{1}{x}\right)^{-1}$$

is positive and continuous on  $[-1, 1]$ . We have  $g(\pm 1) = \frac{1}{2} \exp(-\frac{1}{2})$ . Hence by Properties (ii), (vii) and Theorem 6

$$\lim_{n \rightarrow \infty} \lambda_n(w, x) \cdot n = \pi \sqrt{1-x^2} w(x)$$

uniformly for  $x \in \Delta \subset (-1, 1)$  and

$$\lim_{n \rightarrow \infty} \lambda_n(w, \pm 1) n \exp\left(4\sqrt{\frac{n}{\pi}}\right) = \pi.$$

By Theorem 6.1.27  $w \in M(0, 1)$  since  $w\left(\frac{1}{\pi}\right) \in M(0, 1)$ . By Property 13(vi) and Theorem 6.1.26

$$\lim_{n \rightarrow \infty} \frac{\gamma_n(w)}{2^n n^{\frac{\pi+1}{2\pi}}} = \Gamma\left(\frac{\pi+1}{2\pi}\right) D\left(\frac{w}{\left(\frac{1}{\pi}\right)}, 0\right)^{-1}$$

and by Property 13(ix) and Theorem 6.1.25 for every  $z \in \mathbb{C} \setminus [-1, 1]$

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n(w, z) |\rho(z)|^{2n} \left| \frac{z}{2\pi\sqrt{z^2-1}} \right|^2 &= \\ &= (|\rho(z)|^2-1) \left| \Gamma\left(\frac{1}{2} + \frac{z}{2\pi\sqrt{z^2-1}}\right) \right|^2 \left[ \frac{\sqrt{z^2-1}}{\rho(z)} \right]^{-1 - \frac{z}{\pi\sqrt{z^2-1}}} \\ &\quad \cdot \left| D\left(\frac{w}{\left(\frac{1}{\pi}\right)}, \rho(z)^{-1}\right) \right|^2. \end{aligned}$$

-110-

Further by Theorem 6.1.29 and Property 13(viii) for every  $z \in \mathbb{C} \setminus [-1, 1]$

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n(w, z) \cdot \rho(z)^{-n} \cdot 2\pi \sqrt{1-z^2} &= \\ &= \Gamma\left(\frac{1}{2} + \frac{2\sqrt{z^2-1}}{2\pi\sqrt{z^2-1}}\right)^{-1} \cdot \left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}} + \frac{z}{2\pi\sqrt{z^2-1}} \\ &\cdot D\left(\frac{w}{\left(\frac{1}{\pi}\right)}, \rho(z)^{-1}\right)^{-1}. \end{aligned}$$

Using Example 14 and Theorems 6.1.25-27 we immediately obtain

Theorem 15. Let  $w$  be defined by (II). Let  $g(\geq 0) \in L_w^1$  be equivalent to a strictly positive and Riemann integrable function. Then every result in Example 14 remains true if we replace  $w$  by  $w_g = gw$ , in particular,  $w_g \in M(0, 1)$ .

Now we shall investigate  $G_n(u, f)$  where  $u = u^{(a, b)}$  is a Jacobi weight.

Lemma 16. There exists a constant  $C = C(u)$  such that

$$p_n^2(u, x) \leq C \left[ \sqrt{1-x} + \frac{1}{n} \right]^{-2a-1} \left[ \sqrt{1+x} + \frac{1}{n} \right]^{-2b-1}$$

for  $|x| \leq 1$ ,  $n = 1, 2, \dots$ .

Proof. See Szegő, §7.32.

Lemma 17. For  $m = n-1, n$  and  $n = 1, 2, \dots$

-III-

$$\max_{|x| \leq 1} \lambda_n(u, x) p_m^2(u, x) = O\left(\frac{1}{n}\right).$$

Proof. Apply Lemmas 16 and 6.3.5.

Let us note that Theorem 3.1.11 and Lemma 4 give

$$\max_{x \in \Delta \subset (-1, 1)} \lambda_n(u, x) p_m^2(u, x) = O\left(\frac{1}{n}\right) \quad (m = n-1, n).$$

Lemma 18. Let  $g \in L^1 \cap L_u^1$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{-1}^1 |g(t)| p_n^2(u, t) u(t) dt = 0.$$

Proof. Let us consider  $\int_0^1$ . If  $a < 0$  then use Lemma 16. Let  $a \geq 0$ . Fix  $\varepsilon > 0$ . Then

$$\begin{aligned} \frac{1}{n} \int_0^1 |g(t)| p_n^2(u, t) u(t) dt &\leq \frac{1}{n} \max_{0 \leq t \leq 1-\varepsilon} p_n^2(u, t) \cdot \\ &\cdot \int_{-1}^1 |g(t)| u(t) dt + \int_{1-\varepsilon}^1 |g(t)| dt \end{aligned}$$

again by Lemma 16. First let  $n \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ .

Theorem 19. Let  $g \in L^1 \cap L_u^1$ . If  $g$  is continuous on a closed set  $\mathbb{D} \subset [-1, 1]$  then

$$\lim_{n \rightarrow \infty} G_n(u, g, x) = g(x)$$

uniformly for  $x \in \mathbb{D}$ .

Proof. We have to show that

-III-

$$\lim_{n \rightarrow \infty} \max_{|x| < 1} \{ \lambda_n(u, x) \int_{-1}^1 (x-t)^2 K_n^2(u, x, t) |g(t)| u(t) dt \} = 0$$

and then the machinery of positive operators can be applied. But this is so by Lemmas 17 and 18.

Theorem 20. Let  $g \in L^1 \cap L_u^1$ . Let  $x \in (-1, 1)$  be a Lebesgue point of  $g$ . Then

$$\lim_{n \rightarrow \infty} G_n(u, g, x) = g(x).$$

Proof. Let  $x \in (-1, 1)$  and  $\epsilon > 0$  be fixed. If  $\epsilon$  is small enough then

$$|G_n(u, g, x) - g(x)| \leq$$

$$\lambda_n(u, x) \int_{|x-t| < \frac{1}{n}} + \int_{\frac{1}{n} \leq |x-t| \leq \epsilon} + \int_{|x-t| > \epsilon} |g(t) - g(x)| \cdot K_n^2(u, x, t) u(t) dt.$$

By Lemmas 16-18

$$(12) \quad \lambda_n(u, x) \int_{|x-t| < \frac{1}{n}} < Cn \int_{|x-t| < \frac{1}{n}} |g(t) - g(x)| dt,$$

$$(13) \quad \lambda_n(u, x) \int_{\frac{1}{n} \leq |x-t| \leq \epsilon} \leq \frac{C}{n} \int_{\frac{1}{n} \leq |x-t| \leq \epsilon} \frac{|g(t) - g(x)|}{(x-t)^2} dt$$

and the third term converges to 0 when  $n \rightarrow \infty$ .  $\lim_{n \rightarrow \infty}$  (right side of (12)) = 0 because  $x$  is a Lebesgue point of  $g$ . To estimate the right side of (13) we integrate by parts and remember that  $\epsilon > 0$  is arbitrary.

Lemma 21. If  $a, b > -1$  then

$$\lim_{n \rightarrow \infty} n^{2a+2} \lambda_n(u, 1) = (a+1) 2^{a+b+1} \Gamma(a+1)^2$$

and

$$\lim_{n \rightarrow \infty} n^{2b+2} \lambda_n(u, -1) = (b+1) 2^{a+b+1} \Gamma(b+1)^2.$$

Proof. Szegő, §4.5 and easy calculation.

Theorem 22. Let  $\text{supp}(w) = [-1, 1]$  and suppose that there exists a polynomial  $P$  such that  $w^{-1} P^2 \in L^1(-1, 1)$ . Then for almost every  $x \in [-1, 1]$

$$(14) \quad \lim_{n \rightarrow \infty} \lambda_n(w, x) = \pi \sqrt{1-x^2}.$$

If  $w$  is positive and continuous on a closed set  $\mathbb{R} \subset (-1, 1)$  then (14) holds uniformly for  $x \in \mathbb{R}$ . If  $w$  is continuous at  $x \in (-1, 1)$  then (14) holds. If there exists a Jacobi weight  $u$  such that  $w/u$  is positive and continuous on  $\Delta \subset [-1, 1]$  then

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(w, x)}{\lambda_n(u, x)} = \frac{w(x)}{u(x)}$$

uniformly for  $x \in \Delta$ . If  $w/u$  is positive and continuous at 1 then

$$\lim_{n \rightarrow \infty} n^{2a+2} \lambda_n(w, 1) = \frac{w(1)}{u(1)} (a+1) 2^{a+b+1} \Gamma(a+1)^2.$$

If  $w/u$  is positive and continuous at -1 then

$$\lim_{n \rightarrow \infty} n^{2b+2} \lambda_n(w, -1) = \frac{w(-1)}{u(-1)} (b+1) 2^{a+b+1} \Gamma(b+1)^2.$$



Proof. Put  $g = w$ ,  $\alpha =$  Legendre weight in the first part of the theorem and  $g = w/u$ ,  $\alpha = \int u$  in the second part and use Theorems 3, 5, 19, 20, Lemmas 4, 21 and Example 9.

Corollary 23. Let  $\text{supp}(w) \subset [-1, 1]$ . Then for almost every  $x \in (-1, 1)$

$$\lim_{n \rightarrow \infty} n \lambda_n(w, x) \leq \pi \sqrt{1-x^2} w(x).$$

Proof. Let  $\epsilon > 0$ . Then  $\lambda_n(w, x) \leq \lambda_n(w + \epsilon \chi_{[-1, 1]}^{\epsilon}, x)$  and  $(w + \epsilon \chi_{[-1, 1]}^{\epsilon})^{-1} \in L^1$ .

Corollary 24. If  $\frac{1}{\alpha'} \in L^1(\Delta)$  then

$$\limsup_{n \rightarrow \infty} \frac{1}{n \lambda_n(d\alpha, x)} < \infty$$

for almost every  $x \in \Delta$ .

Proof.  $\lambda_n(d\alpha, x) \geq \lambda_n(\alpha', x) \geq \lambda_n(\alpha')_{\Delta'}$  and transform  $\Delta$  to  $[-1, 1]$ .

In the following we shall improve both corollaries. Corollary 24 is a very strong result. To see this compare Corollary 24 with Freud's result (See Freud, §IV. 6.)

Theorem. Let  $\text{supp}(w) \subset [-1, 1]$  and

$$\int_0^\pi \frac{|w(\cos(\theta+h)) \sin(\theta+h) - w(\cos \theta) \sin \theta|}{w(\cos \theta) \sin \theta} d\theta = O(\log^{-\epsilon} \frac{1}{|h|})$$

for  $h$  small with  $\epsilon > 1$ . Then for almost every  $x \in [-1, 1]$

$$\limsup_{n \rightarrow \infty} \frac{1}{n \lambda_n(w, x)} < \infty.$$

-115-

Let us mention two applications of Corollary 24:

Theorem 25. Let  $\text{supp}(d\alpha) \subset [-1, 1]$  and  $\frac{1}{\alpha'} \in L^1(\Delta)$  where  $\Delta \subset [-1, 1]$ .

If  $f \in L^2_{d\alpha}$  then for almost every  $x \in \Delta$  the Fourier series of  $f$  in  $p_n(d\alpha, x)$  is strongly (C, 1) summable to  $f(x)$ .

Proof. Corollary 24 and Freud, §IV. 3.

Theorem 26. Let  $\alpha \in M(a, b)$  with  $b > 0$  and let  $\frac{1}{\alpha'} \in L^1(\Delta)$  where  $\Delta \subset [a-b, a+b]$ . If

$$\limsup_{n \rightarrow \infty} \sum_{j=n}^{2n} \left\{ \int_{\Delta} \frac{a, b}{c_j} (d\alpha) \right\}^2 < \infty$$

(See Definition 3.1.4.) then the sequence  $\{p_n^2(d\alpha, x)\}$  is bounded for almost every  $x \in \Delta$ .

Proof. Theorem 3.1.11 and Corollary 24.

Later we shall see that in both Theorems 25 and 26 the condition  $1/\alpha' \in L^1(\Delta)$  may be weakened to  $[\alpha']^{-\epsilon} \in L^1(\Delta)$  for some  $\epsilon > 0$ .

Now we shall consider  $G_n(d\alpha, f)$  for weights  $\alpha$  which are less nice than the Jacobi weights. In the following  $\tau, \tau_1$  etc. will denote closed intervals. Recall that  $\tau^0$  denotes the interior of  $\tau$ .

Theorem 27. Let  $\alpha \in M(0, 1)$ ,  $\tau \subset (-1, 1)$ . Let  $\alpha'(t) \geq c > 0$  for almost every  $t \in \tau$ . Let the sequence  $\{p_n^2(d\alpha, t)\}$  be uniformly bounded on every  $\tau_1 \subset \tau^0$ . Let  $f \in L^1_{d\alpha}$  and  $\pi$  be a polynomial vanishing at the endpoints of  $\tau$ . Let  $|f(t) \pi(t)| \leq M < \infty$  for  $d\alpha$  almost every  $t \in \text{supp}(d\alpha) \setminus \tau$ .

Then for almost every  $x \in \tau$

-116-

$$(15) \quad \lim_{n \rightarrow \infty} G_n(d\alpha, f\pi, x) = f(x) \pi(x) .$$

Proof. Let  $x \in \tau^0$  be a  $d\alpha$  Lebesgue point of  $f\pi$ , that is let

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} |f(t) \pi(t) - f(x) \pi(x)| d\alpha(t) = 0 .$$

It is well known that almost every  $x \in \tau^0$  is a  $d\alpha$  Lebesgue point of  $f\pi$ . (See Freud, §IV.2.) First we shall show that

$$(16) \quad \exists \alpha'(x) < \infty \Rightarrow \limsup_{n \rightarrow \infty} n \lambda_n(d\alpha, x) < \infty .$$

Let  $v_\Delta$  be the Tschebyshev weight corresponding to  $\Delta(d\alpha)$ . Then

$$\lambda_n(d\alpha, x) \leq K_n^{-2}(v_\Delta, x, x) \int_{-\infty}^{\infty} K_n^2(v_\Delta, x, t) d\alpha(t) .$$

Since  $x \in \tau^0 \subset \tau \subset \Delta(d\alpha)^0$  we have by Example 8

$$n \lambda_n(d\alpha, x) \leq C \cdot n \left[ \alpha\left(x + \frac{1}{n}\right) - \alpha\left(x - \frac{1}{n}\right) \right] +$$

$$+ \frac{C}{n} \int_{|x-t| \geq \frac{1}{n}} \frac{d\alpha(t)}{(x-t)^2}$$

and integrating the integral by part we obtain (16). Since  $\alpha' \in L^1(-1, 1)$   $\alpha'(x) < \infty$  for almost every  $x \in \tau^0$ . Let now  $x \in \tau^0$  be a  $d\alpha$  Lebesgue point of  $f\pi$  and let  $\alpha'(x) < \infty$ . We shall prove (15) for such points  $x$ . Let  $\varepsilon > 0$  be so small that  $x \pm \varepsilon \in \tau^0$ . If  $n$  is large then

$$\begin{aligned} G_n(d\alpha, f\pi, x) - f(x) \pi(x) &= \\ &= \lambda_n(d\alpha, x) \int_{|x-t| \leq \frac{1}{n}} \frac{1}{n} + \int_{\frac{1}{n} < |x-t| \leq \varepsilon} + \int_{|x-t| > \varepsilon} \\ &= [f(t) \pi(t) - f(x) \pi(x)] K_n^2(d\alpha, x, t) d\alpha(t) . \end{aligned}$$

By (16)

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \int_{|x-t| \leq \frac{1}{n}} = 0 .$$

We have further by (16)

$$\begin{aligned} \lambda_n(d\alpha, x) \int_{\frac{1}{n} < |x-t| \leq \varepsilon} &\leq \frac{C}{n} \int_{\frac{1}{n} < |x-t| \leq \varepsilon} |f(t) \pi(t) - f(x) \pi(x)| \cdot \\ &\cdot \frac{d\alpha(t)}{(x-t)^2} . \end{aligned}$$

Integrating by parts we obtain

$$\limsup_{n \rightarrow \infty} \lambda_n(d\alpha, x) \int_{\frac{1}{n} < |x-t| \leq \varepsilon} \leq$$

$$\leq C \sup_{\frac{1}{h} \leq \varepsilon} \frac{1}{h} \int_x^{x+h} |f(t) \pi(t) - f(x) \pi(x)| d\alpha(t) .$$

Now consider  $\int_{|x-t| > \varepsilon}$ . We have

$$\lambda_n(d\alpha, x) \int_{|x-t| > \varepsilon} \leq \frac{C}{n\varepsilon} \int_{\tau^0} [p_n^2(d\alpha, t) + p_{n-1}^2(d\alpha, t)] \cdot$$

$$[|f(t) \pi(t)| + |f(x) \pi(x)|] d\alpha(t) \leq$$

$$\leq \frac{C}{n^2} |f(x) \pi(x)| + \sum_{k=n-1}^n \frac{C}{n^2} \int_{t \in \tau} p_k^2(d\alpha, t) |f(t) \pi(t)| d\alpha(t).$$

Here we cannot use simple estimates since the sequence  $\{p_k^2(d\alpha, t)\}$  is uniformly bounded only for  $x \in \tau_1 \subset \tau^0$  but not for  $x \in \tau$ . By Theorem 4.1.11

$$\lim_{k \rightarrow \infty} \max_{t \in \tau} \lambda_k(d\alpha, t) p_k^2(d\alpha, t) = 0.$$

Hence for  $k = n-1, n$

$$\begin{aligned} \frac{1}{n} \int_{t \in \tau} p_k^2(d\alpha, t) |f(t) \pi(t)| d\alpha(t) &= \\ &= \sigma(1) \int_{t \in \tau} \frac{1}{k \lambda_k(d\alpha, t)} |f(t) \pi(t)| d\alpha(t). \end{aligned}$$

Let  $v_\tau$  denote the Tschebyshev weight corresponding to  $\tau$ . Then from

$\alpha'(t) \geq c > 0$  for almost every  $t \in \tau$  follows that

$$\lambda_n(d\alpha, t) \geq \frac{C}{n} v_\tau(t)^{-1}$$

for  $t \in \tau$  (See Freud, §III. 3.) Hence for  $k = n-1, n$

$$\begin{aligned} \frac{1}{n} \int_{t \in \tau} p_k^2(d\alpha, t) |f(t) \pi(t)| d\alpha(t) &= \\ &= \sigma(1) \int_{t \in \tau} v_\tau(t) |f(t) \pi(t)| d\alpha(t). \end{aligned}$$

Let us recall that  $\pi$  vanishes at the endpoints of  $\tau$ . Thus

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \int_{t \in \tau} |x-t| > \varepsilon = 0.$$

Finally, by the conditions and (16)

$$\lambda_n(d\alpha, x) \int_{t \in \tau} \frac{1}{t} \leq \frac{C}{n}.$$

Consequently (15) holds for almost every  $x \in \tau$ .

**Theorem 28.** Let  $\alpha \in M(0, 1)$ ,  $\tau \subset (-1, 1)$ ,  $\alpha'(t) \geq c > 0$  for almost every  $t \in \tau$ ,  $\{p_n^2(d\alpha, t)\}$  be uniformly bounded on each  $\tau_1 \subset \tau^0$ . Let  $f \in L_{d\alpha}^1$ ,  $\pi$  be a polynomial vanishing at the endpoints of  $\tau$  and let  $|f(t) \pi(t)| \leq M < \infty$  for  $d\alpha$  almost every  $t \in \text{supp}(d\alpha) \setminus \tau$ . If  $f$  is continuous at  $x \in \tau^0$  and  $|\alpha(x) - \alpha(t)| \leq K|x-t|$  for  $|x-t|$  small then (15) holds. If  $f$  is continuous on  $\tau_1 \subset \tau^0$  and  $\alpha \in \text{Lip } 1$  on  $\tau_2 \setminus (\tau_1 \cup \tau_2^0)$  then (15) is satisfied uniformly for  $x \in \tau_1$ .

Proof. Repeat the proof of Theorem 27 with the necessary modifications.

**Lemma 29.** Let  $\alpha \in S$ . Let  $\tau \subset (-1, 1)$  and  $\alpha$  be absolutely continuous on  $\tau$  with  $\alpha'(t) \equiv 1$  for  $t \in \tau$ . Then the sequence  $\{p_n^2(d\alpha, x)\}$  is uniformly bounded on each  $\tau_1 \subset \tau^0$  and

$$\lim_{n \rightarrow \infty} n \lambda_n(d\alpha, x) = \pi \sqrt{1-x^2}$$

uniformly for  $x \in \tau_1 \subset \tau^0$ . Moreover,  $\alpha \in M(0, 1)$ .

Proof. See Geronimus, §5.4.

**Lemma 30.** Let  $\beta \in S$ . Let  $\tau \in (-1, 1)$  and  $\beta$  be absolutely continuous on  $\tau$  with  $1/\beta' \in L^1(\tau)$ . Then for almost every  $x \in \tau$

$$\lim_{n \rightarrow \infty} n \lambda_n(d\beta, x) = \pi \beta'(x) \sqrt{1-x^2}.$$

**Proof.** Let us define  $\alpha$  and  $g$  by

$$d\alpha(x) = \begin{cases} d\beta(x) & \text{for } x \in [-1, 1] \setminus \tau \\ dx & \text{for } x \in \tau \end{cases}$$

with  $\text{supp}(d\alpha) = [-1, 1]$  and

$$g(x) = \begin{cases} 1 & \text{for } x \in [-1, 1] \setminus \tau \\ \beta'(x) & \text{for } x \in \tau. \end{cases}$$

Then  $\alpha \in S$ ,  $g(x)^{+1} = 1 < \infty$  for  $x \in [-1, 1] \setminus \tau$  and  $\beta = \alpha_g$ . Further  $\alpha$  satisfies the conditions of Lemma 29 and consequently  $\alpha$  satisfies also the conditions of Theorem 27. Let us put  $P_1 = v_\tau^{-2}$  where  $v_\tau$  is the Tschebyshev weight corresponding to  $\tau$ . Then by Theorem 5

$$\frac{v_\tau^{-4}(x) \lambda_n(d\beta, x)}{\lambda_{n-2}(d\alpha, x)} \leq G_{n-2}(d\alpha, g v_\tau^{-4}, x).$$

Since  $v_\tau^{-4}$  vanishes at the endpoints of  $\tau$  we obtain from Lemmas 4, 29 and Theorem 27

$$\limsup_{n \rightarrow \infty} n \lambda_n(d\beta, x) \leq \pi \sqrt{1-x^2} \beta'(x)$$

for almost every  $x \in \tau$ .

On the other hand putting  $P_2 = v_\tau^{-2}$  and using Theorem 5

$$v_\tau^{-4}(x) G_{n+2}^{-1}(d\alpha, g^{-1} v_\tau^{-4}) \leq \frac{\lambda_n(d\beta, x)}{\lambda_{n+2}(d\alpha, x)}.$$

Thus by the same arguments  $(g^{-1} \in L_{d\alpha}^1)$

$$\pi \sqrt{1-x^2} \beta'(x) \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n(d\beta, x)}{\lambda_n(d\alpha, x)}.$$

**Lemma 31.** Let  $\alpha$  be an arbitrary weight. Then for almost every  $x \in [-1, 1]$

$$\lim_{n \rightarrow \infty} \lambda_n(v, x) \int_{-1}^1 K_n^2(v, x, t) d[\alpha_g(t) + \alpha_f(t)] = 0.$$

**Proof.** We have for almost every  $x \in [-1, 1]$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \int_{s-h}^{s+h} |d[\alpha_s(t) + \alpha_f(t)]| &= \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{s-h}^{s+h} d[\alpha_s(t) + \alpha_f(t)] = 0 \end{aligned}$$

and we can use standard argument.

**Lemma 32.** Let  $\alpha$  be an arbitrary weight. Then for almost every  $x \in [-1, 1]$

$$\lim_{n \rightarrow \infty} \lambda_n(v, x) \int_{-1}^1 K_n^2(v, x, t) d\alpha(t) = \sqrt{1-x^2} \alpha'(x).$$

**Proof.** Since  $d\alpha(t) = \alpha'_s(t) dt + d[\alpha_s(t) + \alpha_f(t)] = g(t) v(t) dt + d[\alpha_s(t) + \alpha_f(t)]$  where  $g = \alpha_s v^{-1}$  the lemma follows from Theorem 20 and Lemma 31.

**Theorem 33.** If  $\text{supp}(d\alpha) \subset [-1, 1]$  then



$$\limsup_{n \rightarrow \infty} n \lambda_n(d\alpha, x) \leq \pi \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in [-1, 1]$ .

Proof. Use Lemma 32 and the inequality

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v, x)} \leq \lambda_n(v, x) \int_{-1}^1 K_n^2(v, x, t) d\alpha(t)$$

which follows from  $\text{supp}(d\alpha) \subset [-1, 1]$ .

Now we can prove the following

Theorem 34. Let  $\alpha \in S$ ,  $\tau \in [-1, 1]$ . If  $1/\alpha' \in L^1(\tau)$  then

$$\lim_{n \rightarrow \infty} n \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in \tau$ .

Proof. By Theorem 33 we have to show that

$$(17) \quad \liminf_{n \rightarrow \infty} n \lambda_n(d\alpha, x) \geq \pi \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in \tau$ . We can assume  $\tau \subset (-1, 1)$ . Since

$$n \lambda_n(d\alpha, x) \geq n \lambda_n(\alpha', x)$$

and  $\alpha'$  satisfies the conditions of Lemma 30 (17) follows from Lemma 30.

Theorem 35. Let  $\alpha \in S$ . Let  $x \in (-1, 1)$ , let  $\alpha$  be absolutely continuous near  $x$  and let  $\alpha'$  be continuous at  $x$ . Then

$$(18) \quad \lim_{n \rightarrow \infty} n \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2}.$$

If  $\alpha$  is absolutely continuous on  $\tau \subset (-1, 1)$ ,  $\alpha'$  is continuous and positive on  $\tau_1 \subset \tau^0$  then (18) holds uniformly for  $x \in \tau_1$ .

Proof. The proof is the same as that of Lemma 30. If in the first part of the theorem  $\alpha'(x) = 0$  then we fix  $\varepsilon > 0$  and we prove first that

$$\lim_{n \rightarrow \infty} n \lambda_n(d\beta, x) = \pi \beta'(x) \sqrt{1-x^2}$$

where  $\beta(x) = \alpha(x) + \varepsilon x$  with  $\text{supp}(d\beta) = [-1, 1]$  and after we let  $\varepsilon \rightarrow 0$ .

Let us note that the first part of Theorem 35 easily follows from results of Geronimus and the second part of it has been obtained by Geronimus in his book. Further, we would not have obtained a stronger result in Theorem 34 if we had supposed the  $\pi^2/\alpha' \in L^1(\tau)$  with a suitable polynomial  $\pi$ . Theorem 34 can be formulated as

Theorem 36. Let  $\alpha \in S$ ,  $\tau \subset [-1, 1]$ . If  $1/\alpha' \in L^1(\tau)$  then

$$(C, 1) \lim_{n \rightarrow \infty} [\alpha'(x) \sqrt{1-x^2}]^2 p_n^2(d\alpha, x) - \frac{2}{\pi} \cos^2(n\theta - \Gamma(\theta)) = 0$$

for almost every  $x \in \tau$ , where  $x = \cos\theta$  ( $0 \leq \theta \leq \pi$ ) and  $\Gamma(\theta)$  is defined in Definition 4.2.4.

Later we shall improve Theorem 36 for weights having nice coefficients in the recursion formula.

Definition 37. Let  $\omega$  be a modulus of continuity. Then

$$(t) \quad f \in A_x^\omega \text{ iff } |f(x) - f(t)| \leq C_x \omega(|x-t|)$$

for  $|x-t|$  small.

(ii)  $f \in A_\tau^\omega$  if  $|f(x) - f(t)| \leq C_\tau \omega(|x-t|)$

for  $x \in \tau$ ,  $t \in \tau(\epsilon)$  where  $\tau(\epsilon)$  is an  $\epsilon$  neighborhood of  $\tau$ .

(iii)  $f \in B_x^\omega$  iff  $f'(x)$  exists and

$$|f(t) - f(x) - f'(x)(t-x)| \leq C_x \omega(|t-x|) |t-x|$$

for  $|x-t|$  small.

(iv)  $f \in B_\tau^\omega$  if  $f'(x)$  exists for  $x \in \tau$  and

$$|f(t) - f(x) - f'(x)(t-x)| \leq C_\tau \omega(|t-x|) |t-x|$$

for  $x \in \tau$  and  $t \in \tau(\epsilon)$ .

Theorem 38. Let  $\text{supp}(d\sigma)$  be compact,  $x \in \text{supp}(d\sigma)$ ,  $\sigma$  be absolutely continuous near  $x$ ,  $0 < c_1 \leq \sigma'(t) \leq c_2 < \infty$  for  $|x-t|$  small and let the sequence  $\{p_n^2(d\sigma, t)\}$  be uniformly bounded for  $|x-t|$  small. Let  $f$  be  $d\sigma$  measurable and bounded on  $\text{supp}(d\sigma)$ . If  $f \in A_x^\omega$  then

$$(19) \quad G_n(d\sigma, f, x) = f(x) + O(1) \frac{1}{n} \int_1^{\frac{1}{t}} \frac{\omega(t)}{t^2} dt,$$

where  $|O(1)| < C$  with  $C$  independent of  $n$ . If  $\text{supp}(d\sigma)$  is compact,  $\sigma$  is absolutely continuous in  $\tau(\epsilon)$ ,  $0 < c_1 \leq \sigma'(t) \leq c_2 < \infty$  in  $\tau(\epsilon)$ , the sequence  $\{p_n^2(d\sigma, t)\}$  is uniformly bounded in a neighborhood of  $\tau$ ,  $f$  is  $d\sigma$  measurable and bounded on  $\text{supp}(d\sigma)$  and  $f \in A_\tau^\omega$  then (19) holds uniformly for  $x \in \tau$ .

Proof. Let us prove, for simplicity, the first part of the theorem. Let  $\epsilon > 0$  be small. Then

$$\begin{aligned} |G_n(d\sigma, f, x) - f(x)| &\leq \\ &\leq \frac{C}{n} \left\{ \int_{|x-t| < \epsilon} |f(t) - f(x)| \frac{1}{t^2} dt + \int_{|x-t| > \epsilon} |f(t) - f(x)| K_n^2(d\sigma, x, t) d\sigma(t) \right\}. \end{aligned}$$

-125-

The second integral is  $O(1)$ , the first one is

$$O(1) \int_{|x-t| < \epsilon} \omega(|t-x|) \frac{n^2}{(1+n|t-x|)^2} dt$$

by the usual computation and the latter integral can easily be estimated.

Lemma 39. For  $n = 1, 2, \dots$

$$\begin{aligned} \lambda_n(d\sigma, x) \int_{-\infty}^{\infty} (t-x) K_n^2(d\sigma, x, t) d\sigma(t) &= \\ &= - \frac{Y_{n-1}(d\sigma)}{Y_n(d\sigma)} \lambda_n(d\sigma, x) p_{n-1}(d\sigma, x) p_n(d\sigma, x). \end{aligned}$$

Proof. See Freud, §V.6.

Theorem 40. Let the conditions of the first part of Theorem 38 be satisfied with  $f \in B_x^\omega$  instead of  $f \in A_x^\omega$ . Then

$$(20) \quad G_n(d\sigma, f, x) = f(x) + O(1) \frac{1}{n} \int_1^{\frac{1}{t}} \frac{\omega(t)}{t} dt$$

where  $|O(1)| \leq C$  and  $C$  does not depend on  $n$ . If the conditions of the second part of Theorem 38 are satisfied with  $f \in B_\tau^\omega$  instead of  $f \in A_\tau^\omega$  then (20) holds uniformly for  $x \in \tau$ .

Proof. For simplicity let us consider the first part of the theorem. From the proof of Theorem 38 we see that we have to show

$$\left| \int_{|x-t| < \epsilon} [f(t) - f(x)] K_n^2(d\sigma, x, t) d\sigma(t) \right| = O(1) \int_1^{\frac{1}{t}} \frac{\omega(t)}{t} dt.$$

The left side here is not greater than

-126-

**Definition 42.** For a given weight  $\beta$  and interval  $\tau$  the weight  $\beta_\tau$  and the function  $g \equiv g_{\beta, \tau}$  are defined by

$$d\beta_\tau(x) = \begin{cases} d\beta(x) & \text{for } x \in \tau \\ dx & \text{for } x \notin \tau \end{cases}$$

and

$$g(x) = \begin{cases} \beta'(x) & \text{for } x \in \tau \\ 1 & \text{for } x \notin \tau \end{cases}.$$

We have  $d(\beta_\tau)g \leq d\beta$  and equality holds iff  $\beta$  is absolutely continuous on  $\tau$ .

**Theorem 43.** Let  $\beta \in S$ ,  $x \in (-1, 1)$ ,  $\beta$  be absolutely continuous near  $x$ ,  $\beta' \in A_x^\omega$  (or  $\beta' \in B_x^\omega$ ),  $\beta'(x) > 0$ ,  $\tau^0$  be a sufficiently small neighborhood of  $x$ . Then

$$\frac{\lambda_n(d\beta, x)}{\lambda_n(d\beta_\tau, x)} = \beta'(x) + O(1) \begin{cases} \frac{1}{n} \int_1^{\frac{1}{n}} \frac{\omega(t)}{t^2} dt & (\beta' \in A_x^\omega) \\ \frac{1}{n} \int_1^{\frac{1}{n}} \frac{\omega(t)}{t} dt & (\beta' \in B_x^\omega) \end{cases}.$$

If  $\tau_1 \subset (-1, 1)$ ,  $\beta$  is absolutely continuous in  $\tau_1(\epsilon)$  with some  $\epsilon > 0$ ,  $\beta' \in A_{\tau_1}^\omega$  (or  $\beta' \in B_{\tau_1}^\omega$ ),  $\tau^0$  is a sufficiently small neighborhood of  $\tau_1$  then

-128-

$$\int_{|x-t|<\epsilon} |f(t) - f(x) - f'(x)(t-x)| K_n^2(d\alpha, x, t) d\alpha(t) +$$

$$+ |f'(x)| \int_{|x-t|<\epsilon} (t-x) K_n^2(d\alpha, x, t) d\alpha(t).$$

Since  $f \in B_x^\omega$  the first integral here may easily be estimated. Further

$$\int_{|x-t|<\epsilon} (t-x) K_n^2(d\alpha, x, t) d\alpha(t) = \left\{ \int_{-\infty}^{\infty} - \int_{|x-t| \geq \epsilon} \right\}$$

$$(t-x) K_n^2(d\alpha, x, t) d\alpha(t)$$

and we apply Lemma 39.

**Remark 41.** If  $f \in A_x^\omega$  and  $f(x) > 0$  then  $f^{-1} \in A_x^\omega$ . If  $f \in A_\tau^\omega$  and  $f(x) > 0$  for  $x \in \tau$  then  $f^{-1} \in A_\tau^\omega$ . This is obvious. If  $f \in B_x^\omega$  and  $f(x) > 0$  then  $f^{-1} \in B_x^\omega$  and if  $f \in B_\tau^\omega$  and  $f(x) > 0$  for  $x \in \tau$  then  $f^{-1} \in B_\tau^\omega$ . Let us prove the latter. Because of continuity  $f(t) > 0$  for  $|x-t|$  small and  $f(t) > 0$  for  $t \in \tau(\epsilon)$  with  $\epsilon$  small respectively. Now we can use the identity

$$\frac{1}{f(t)} - \frac{1}{f(x)} + \frac{f'(x)}{f^2(x)}(t-x) = \frac{[f(x)-f(t)]^2}{f^2(x)f(t)} -$$

$$- \frac{1}{f^2(x)} [f(t) - f(x) - f'(x)(t-x)]$$

and

$$[f(x) - f(t)]^2 \leq C(t-x)^2 \leq C|t-x| \omega(|t-x|)$$

where  $C$ , of course, depends on  $f$ .

-127-



$$\frac{\lambda_n(d\beta, x)}{\beta_n(d\beta, x)} = \beta'(x) + O(1) \begin{cases} \frac{1}{n} \int_{-1}^1 \frac{\omega(t)}{t^2} dt & (\beta' \in A_{\tau_1}^{\omega}) \\ \frac{1}{n} \int_{-1}^1 \frac{\omega(t)}{t} dt & (\beta' \in B_{\tau_1}^{\omega}) \end{cases}$$

uniformly for  $x \in \tau_1$ .

Proof. If  $\tau$  is small then  $d(\beta_{\tau})g = d\beta$ ,  $g^{\sharp l}$  is bounded on  $[-1, 1]$ ,  $g^{\sharp l} \in A_X^{\omega}(B_X^{\omega})$  or  $g^{\sharp l} \in A_{\tau_1}^{\omega}(B_{\tau_1}^{\omega})$  respectively by Remark 41. Further  $\beta_{\tau}$  satisfies the conditions of Lemma 29 and consequently  $\beta_{\tau}$  satisfies the conditions of Theorems 38 and 40. Finally, apply Theorem 3.

Lemma 44. If  $\tau \in [-1, 1]$  then

$$\lambda_n(v, x) \int_{-1}^1 K_n^2(v, x, t) d\beta_{\tau}(t) \leq \sqrt{1-x^2} + O\left(\frac{1}{n}\right)$$

uniformly for  $x \in \tau_1 \subset \tau^0$  and consequently if  $\text{supp}(d\beta) \subset [-1, 1]$  and

$$n \lambda_n(d\beta_{\tau}, x) \leq \sqrt{1-x^2} + O\left(\frac{1}{n}\right)$$

uniformly for  $x \in \tau_1 \subset \tau^0$ .

Proof. See Freud, §V.6.

Lemma 45. Let  $\text{supp}(d\beta) \subset [-1, 1]$ ,  $\tau \subset (-1, 1)$ . Let exist a polynomial  $\pi$  such that  $\pi^2/\beta' \in L^1(-1, 1)$ . Then

$$\frac{1}{n \lambda_n(d\beta_{\tau}, x)} \leq \frac{1}{\pi \sqrt{1-x^2}} + O\left(\frac{1}{n}\right)$$

uniformly for  $x \in \tau_1 \subset \tau^0$ .

Proof. Let us consider (7) in Theorem 5. We put there  $\alpha = \text{Tschébyshév}$  weight,  $g = \beta'_1/v$  so that  $d\alpha_g(x) = \beta'_1(x)dx$ . Let  $P_2 = v^{-2}\pi$ . We obtain

$$\frac{v^{-4}(x) \pi^2(x) \lambda_{n+m}(v, x)}{\lambda_n(\beta'_1, x)} \leq G_{n+m}\left(v, \frac{v^{-3/2}}{\beta'_1}, x\right)$$

where  $m = \deg \pi + 2$  and  $v^{-3/2}/\beta'_1 \in L^1_v$ . Since  $\beta'_1(t) = 1$  for  $t \in \tau$  we can suppose that  $\pi$  has no zeros in  $\tau^0$ . Hence for  $x \in \tau^0$

$$\lambda_n^{-1}(d\beta_{\tau}, x) \leq \pi^{-2}(x) v^{-4}(x) \lambda_{n+m}^{-1}(v, x) G_{n+m}\left(\frac{v^{-3/2}}{\beta'_1}, x\right).$$

Now we should apply Theorem 40 with  $\alpha = \text{Tschébyshév}$  weight,

$f = v^{-3/2}/\beta'_1 \in B_{\tau_1}^{\omega}$  if  $\tau_1 \subset \tau^0$  and  $\omega(t) \equiv t$ , but we cannot do this directly since in our case  $f$  is not bounded on  $\text{supp}(v)$ . This small

problem can be avoided by remarking that the Tschébyshév polynomials are uniformly bounded on  $[-1, 1]$  and thus

$$\int_{|x-t|>\varepsilon} |f(t)| K_n^2(v, x, t) v(t) dt \leq \frac{C}{\varepsilon} \int_{-1}^1 |f(t)| v(t) dt$$

which in our case is finite. Hence

$$G_{n+m}\left(v, \frac{v^{-3/2}}{\beta'_1}, x\right) = \frac{v^{-3}(x) \pi^2(x)}{\beta'_1(x)} + O\left(\frac{1}{n}\right)$$

uniformly for  $x \in \tau_1 \subset \tau^0$  which proves the lemma.



Lemmas 44 and 45 give us

Theorem 46. Let  $\text{supp}(d\alpha) \subset [-1, 1]$ ,  $\tau \in (-1, 1)$  and let exist a polynomial

$\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Then

$$n \lambda_n(d\alpha, x) = \pi \sqrt{1-x^2} + O\left(\frac{1}{n}\right)$$

uniformly for  $x \in \tau_1 \subset \tau$ .

We obtain immediately from Theorems 43 and 46 the following

Theorem 47. Let  $\text{supp}(d\alpha) = [-1, 1]$  and suppose that there exists a poly-

nomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . If  $x \in (-1, 1)$ ,  $\alpha$  is absolutely

continuous near  $x$ ,  $\alpha' \in A_X^\omega(B_X^\omega)$  and  $\alpha'(x) > 0$  then

$$n \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2} + O(1) \quad \left\{ \begin{array}{l} \frac{1}{n} \int_1^{\frac{1}{\alpha'(x)}} \frac{\omega(t)}{t^2} dt \\ \frac{1}{n} \int_1^{\frac{1}{\alpha'(x)}} \frac{\omega(t)}{t} dt \end{array} \right. \quad \begin{array}{l} (\alpha' \in A_X^\omega) \\ (\alpha' \in B_X^\omega) \end{array}$$

If  $\tau \subset (-1, 1)$ ,  $\alpha$  is absolutely continuous in a neighborhood of  $\tau$ ,

$\alpha' \in A_\tau^\omega(B_\tau^\omega)$  and  $\alpha'(t) > 0$  for  $t \in \tau$  then

$$n \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2} + O(1) \quad \left\{ \begin{array}{l} \frac{1}{n} \int_1^{\frac{1}{\alpha'(x)}} \frac{\omega(t)}{t^2} dt \\ \frac{1}{n} \int_1^{\frac{1}{\alpha'(x)}} \frac{\omega(t)}{t} dt \end{array} \right. \quad \begin{array}{l} (\alpha' \in A_\tau^\omega) \\ (\alpha' \in B_\tau^\omega) \end{array}$$

The reader should compare Theorem 47 with Freud's results where

$\pi^2/\alpha' \in L^\infty$  has to be assumed. (See Freud, §V.6).

In Theorem 40 we have shown that  $G_n(d\alpha, f)$  will converge to  $f$  with speed  $1/n$  if  $f$  is good. On the other hand for  $f \in \text{Lip } 1$  we have only obtained  $\log n/n$  as convergence speed for  $G_n(d\alpha, f)$ . (See Theorem 38.) We may ask two questions, namely, whether  $\log n/n$  occurs because of our weak techniques and how to improve convergence.

Theorem 48. Let  $f(x) \equiv |x|$ . Then

$$G_n(v, f, 0) \geq C \cdot \frac{\log n}{n}$$

for  $n \geq 3$ .

Proof. Since

$$G_n(v, f, 0) = 2 \lambda_n(v, 0) \int_0^1 t K_n^2(v, 0, t) v(t) dt$$

we have only to show that for  $k$  odd

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 kt}{\cos t} dt \geq C \log k \quad (k \geq 3).$$

The left side here equals

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{\sin t} dt \geq \int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{t} dt = \int_0^{\frac{k\pi}{2}} \frac{\sin^2 u}{u} du \geq C \log k.$$

By Theorem 48 if we want to improve the convergence properties of  $G_n(d\alpha, f)$  then we have to modify these operators. Let us put for  $n \leq m$

$$G_{n,m}(d\alpha, f, x) = \lambda_n(d\alpha, x) \int_{-\infty}^{\infty} f(t) K_n(d\alpha, x, t) K_m(d\alpha, x, t) d\alpha(t).$$

For  $z \in \mathbb{C}$   $G_{n,m}(d\alpha, f, z)$  can be defined by

$$G_{n,m}(d\alpha, f, z) = \lambda_n^*(d\alpha, z) \int_{-\infty}^{\infty} f(t) k_n(d\alpha, z, t) k_m(d\alpha, z, t) d\alpha(t).$$

(See 4.1).

Properties 49. (i)  $G_{n,m}(d\alpha, \pi_{m-n}) \equiv \pi_{m-n}$ . (ii)  $G_{n,m}$  is a rational function of degree  $(n+m-2, 2n-2)$ . (iii) The Lebesgue function  $G_{n,m}^*(d\alpha, x)$  of the operator  $G_{n,m}(d\alpha)$  is not greater than  $[\lambda_n(d\alpha, x) \lambda_m^{-1}(d\alpha, x)]^{\frac{1}{2}}$ .

Consequently if  $f$  is good globally and  $\alpha$  is nice locally (near

$x \in \text{supp}(d\alpha)$ ) then for e.g.  $m = 2n$   $G_{n,2n}(d\alpha, f, x)$  may converge to  $f(x)$

very rapidly. On the other hand if  $\alpha$  is nice near  $x$  then the kernel

function of  $G_{n,2n}(d\alpha)$  has the same majorant only as that of

$G_n(d\alpha) \equiv G_{n,n}(d\alpha)$ , namely

$$(21) \quad \frac{Cn}{2^{1+n} (x-t)^2},$$

which - as is well known - is too weak to assume good convergence

properties for  $G_{n,2n}(d\alpha, f, x)$  if  $f$  is nice only at  $x$ . For this reason

we introduce another operator  $G_N(d\alpha)$  ( $N = (n_1, n_2, \dots, n_k)$ ).

Let  $k \geq 2$  be fixed and let  $n_1 \leq n_2, n_1-1 + n_2-1 \leq n_3-1$  and, in general,  $\sum_{j=1}^{i-1} (n_j-1) \leq n_i-1$  for  $i = 2, \dots, k$ . We put

$$G_N(d\alpha, f, x) = \prod_{i=1}^{k-1} \lambda_{n_i}(d\alpha, x) \int_{-\infty}^{\infty} f(t) \prod_{i=1}^k K_{n_i}(d\alpha, x, t) d\alpha(t)$$

and for  $z \in \mathbb{C}$  we define  $G_N(d\alpha, f, z)$  in exactly the same way as we did

when  $k = 2$ .

Let us note that if e.g.  $\alpha =$  Ischebyshev weight then the kernel function of  $G_N(d\alpha)$  may be majorated by

$$(22) \quad \frac{Cn}{1+n^k |x-t|^k}$$

$(x, t \in [-1, 1])$  and all  $n_i$  are of order  $n$ , which differs very much from

(21)! (22) implies that for  $k \geq 3$   $G_N(v, f, x)$  converges to  $f(x)$  with speed  $1/n$  if all  $n_i$  are of order  $n$  and  $f \in A_x^\omega$  with  $\omega(t) \equiv t$ .

It should be possible to improve most of the results of this section by using  $G_N(d\alpha)$  instead of  $G_n(d\alpha)$ . At the present time we cannot do this. Let us mention a simple result which we shall need later.

Lemma 50. Let  $x \in \mathbb{R}$ . Then

$$|\pi_n(x)| \leq \frac{1}{2} \lambda_n(d\alpha, x) \int_{-\infty}^{\infty} |\pi_n(t)| [K_n^2(d\alpha, x, t) + K_{2n}^2(d\alpha, x, t)] d\alpha(t).$$

Proof. Use Property 49(i).

Theorem 51. Let  $\alpha \in M(0, 1)$ . Then

$$(23) \quad \limsup_{n \rightarrow \infty} \lambda_n(d\alpha, x) \leq \pi_{\alpha'}(x) \sqrt{1-x^2}$$

for almost every  $x \in \text{supp}(d\alpha)$ .

Proof. By Theorem 3.3.7 it is enough to show that (23) holds for almost every  $x \in [-1, 1]$ . If  $\text{supp}(d\alpha) = [-1, 1]$  then (23) follows from Theorem 53.

Let now  $\Delta = \text{supp}(d\alpha) \setminus [-1, 1]$  be not empty. Then there exists  $\varepsilon_1 > 0$

such that for every  $\epsilon \in (0, \epsilon_1)$   $\Delta_\epsilon = \text{supp}(d\alpha) \setminus [-1-\epsilon, 1+\epsilon]$  is not empty.

By Theorem 3.3.7  $\Delta_\epsilon$  contains finitely many points. Let  $\Delta_\epsilon = \{a_k\}_{k=1}^m$  where  $m = m(\epsilon)$ . Let  $\pi$  be defined by

$$\pi(x) = \prod_{k=1}^m (x - a_k).$$

Then for  $n > m$

$$\lambda_n(d\alpha, x) \leq \int_{-\infty}^{\infty} \frac{K_{n-m}^2(v, x, t) \pi^2(t)}{K_{n-m}^2(v, x, x) \pi^2(x)} d\alpha(t)$$

for  $x \in [-1, 1]$  where  $v_2$  denotes the Tscheyshchev weight corresponding to  $[-1-\epsilon, 1+\epsilon]$ . Since  $\pi$  vanishes on  $\Delta_\epsilon$  we obtain

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v, x)} \leq \frac{\lambda_{n-m}(v, x)}{\lambda_{n-m}(v, x)} \pi(x)^{-2}.$$

$$\lambda_{n-m}(v, x) \int_{-1-\epsilon}^{1+\epsilon} K_{n-m}^2(v, x, t) \pi^2(t) d\alpha(t).$$

Transforming  $[-1-\epsilon, 1+\epsilon]$  into  $[-1, 1]$  we get

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v, \frac{x}{1+\epsilon})} \leq \frac{\lambda_{n-m}(v, \frac{x}{1+\epsilon})}{\lambda_{n-m}(v, \frac{x}{1+\epsilon})} \pi(x)^{-2}.$$

$$\lambda_{n-m}(v, \frac{x}{1+\epsilon}) \int_{-1}^1 K_{n-m}^2(v, \frac{x}{1+\epsilon}, t) \pi^2((1+\epsilon)t) d\alpha((1+\epsilon)t).$$

Let  $\beta$  be defined by  $d\beta(t) = \pi^2((1+\epsilon)t) d\alpha((1+\epsilon)t)$ . Then  $\beta$  is a weight on  $[-1, 1]$ . Hence by Lemmas 4 and 32

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n(d\alpha, x)}{\lambda_n(v, \frac{x}{1+\epsilon})} \leq \sqrt{1 - \frac{x^2}{(1+\epsilon)^2}} \pi(x)^{-2} \beta\left(\frac{x}{1+\epsilon}\right) = \sqrt{(1+\epsilon)^2 - x^2} \alpha'(x)$$

for almost every  $x \in [-1, 1]$ , that is by Example 9

$$\limsup_{n \rightarrow \infty} \lambda_n(d\alpha, x) \leq \pi \alpha'(x) \sqrt{(1+\epsilon)^2 - x^2}$$

for almost every  $x \in [-1, 1]$ . Now let  $\epsilon \rightarrow 0$ .

Now we can prove the following theorem which is one of our main results.

**Theorem 52.** Let  $\alpha \in M(0, 1)$ ,  $\tau \in (-1, 1)$ ,  $1/\alpha' \in L^1(\tau)$ . Let exist a sequence  $\{\epsilon_k\}_{k=1}^\infty$  with  $\lim_{k \rightarrow \infty} \epsilon_k = 0$  such that  $\log \alpha'(t)/\sqrt{(1-\epsilon_k)^2 - x^2} \in L^1(-1+\epsilon_k, 1-\epsilon_k)$  for every fixed  $k$ . Then

$$(24) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in \tau$ .

**Proof.** Because of Theorem 51 we only have to show that

$$\liminf_{n \rightarrow \infty} \lambda_n(d\alpha, x) \geq \pi \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in \tau$ . We have  $\lambda_n(d\alpha, x) \geq \lambda_n(\alpha', x)$ . Let  $k$  be fixed and  $w$  be defined by  $w(t) = \alpha'(1-\epsilon_k)t$  for  $-1 \leq t \leq 1$  with  $\text{supp}(w) = [-1, 1]$ . By the conditions,  $w \in S$  and  $w^{-1} \in L^1(\tau_1)$ , where  $\tau_1 = [(1-\epsilon_k)^{-1}c_1, (1-\epsilon_k)^{-1}c_2]$  if  $\tau = [c_1, c_2]$ . By Theorem 34

$$\lim_{n \rightarrow \infty} n \lambda_n(w, x) = \pi \sqrt{1-x^2} w(x)$$

for almost every  $x \in \tau_1$ . We have by Example 10

$$\begin{aligned} \lambda_n(w, x) &= (1-\epsilon_k)^{-1} \lambda_n(l, \alpha', (1-\epsilon_k)x) \\ &\leq (1-\epsilon_k)^{-1} \lambda_n(\alpha', (1-\epsilon_k)x) \end{aligned}$$

where  $\Delta = [-1+\epsilon_k, 1-\epsilon_k]$ . Hence

$$\liminf_{n \rightarrow \infty} n \lambda_n(\alpha', (1-\epsilon_k)x) \geq \pi \sqrt{(1-\epsilon_k)^2 - (1-\epsilon_k)^2} x \cdot \alpha'((1-\epsilon_k)x)$$

for almost every  $x \in \tau_1$ , that is

$$\liminf_{n \rightarrow \infty} n \lambda_n(\alpha', x) \geq \pi \sqrt{(1-\epsilon_k)^2 - x^2} \alpha'(x)$$

for almost every  $x \in \tau$ . Now let  $k \rightarrow \infty$ .

Later we shall show that if e.g.

$$\sum_{j=0}^{\infty} \left\{ |a_j(d\alpha)| + \left| \frac{y_j(d\alpha)}{y_{j+1}(d\alpha)} - \frac{1}{2} \right| \right\} < \infty$$

then for each  $\tau \subset (-1, 1)$  all the conditions of Theorem 52 are satisfied

and thus (24) holds for almost every  $x \in [-1, 1]$ . Let us note that Theorem 34 follows from Theorem 52.

Corollary 53. Let  $\alpha$  and  $\tau$  satisfy the conditions of Theorem 52 and

let  $l$  be a fixed nonnegative integer. Then for almost every  $x \in \tau$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+l}(d\alpha, x) = \frac{T_l(x)}{\pi \alpha'(x) \sqrt{1-x^2}}$$

where  $T_l$  denotes the  $l$ -th Tschebyshev polynomial.

Proof. Use Theorems 52 and 4.1.19.

Theorem 54. Let  $\alpha \in M(0, 1)$ . Then for almost every  $x \in \text{supp}(d\alpha)$

$$(25) \quad \limsup_{n \rightarrow \infty} n \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1-x^2}.$$

Proof. Let  $\mathbb{R}$  be defined by  $\mathbb{R} = \{x: \alpha'(x) > 0\}$ . By Theorem 33 we have to show that (25) holds for almost every  $x \in \mathbb{R}$ .

Since  $\alpha$  is almost everywhere continuous, for every  $x \in (-1, 1)$  we can find a sequence  $\{\epsilon_m\}$  such that  $\epsilon_m > 0$ ,  $\lim_{m \rightarrow 0} \epsilon_m = 0$ ,  $[x - \epsilon_m, x + \epsilon_m] \subset (-1, 1)$  and  $\alpha$  is continuous at  $x - \epsilon_m$  and  $x + \epsilon_m$ . By Theorem 4.2.14

$$\lim_{n \rightarrow \infty} \int_{x-\epsilon_m}^{x+\epsilon_m} \frac{1}{n \lambda_n(d\alpha, t)} d\alpha(t) = \frac{1}{\pi} \int_{x-\epsilon_m}^{x+\epsilon_m} \frac{dt}{\sqrt{1-t^2}}$$

for  $m = 1, 2, \dots$ . Thus by Fatou's lemma

$$\frac{1}{2\epsilon_m} \int_{x-\epsilon_m}^{x+\epsilon_m} \liminf_{n \rightarrow \infty} \frac{1}{n \lambda_n(d\alpha, t)} \alpha'(t) dt \leq \frac{1}{2\pi\epsilon_m} \int_{x-\epsilon_m}^{x+\epsilon_m} \frac{dt}{\sqrt{1-t^2}}.$$

Letting  $m \rightarrow \infty$  and using Lebesgue's theorem we obtain

$$\lim_{n \rightarrow \infty} \inf \frac{1}{n \lambda_n(d\alpha, x)} \alpha'(x) \leq \frac{1}{\pi \sqrt{1-x^2}}$$



for almost every  $x \in [-1, 1]$ . By Theorem 3.3.7  $\mathfrak{A} \subset [-1, 1]$ . Hence for almost every  $x \in \mathfrak{A}$

$$\limsup_{n \rightarrow \infty} \lambda_n(d\alpha, x) \geq \pi \alpha'(x) \sqrt{1-x^2}.$$

The converse inequality has been proved in Theorem 3.3.

Theorem 55. Let  $\text{supp}(d\alpha) = [-1, 1]$  and  $\alpha'(x) > 0$  for almost every  $x \in [-1, 1]$ . Then (25) holds for almost every  $x \in [-1, 1]$ .

Proof. If  $f$  is continuous on  $[-1, 1]$  then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(x_k) \lambda_n(d\alpha) = \frac{1}{\pi} \int_{-1}^1 f(t) \frac{dt}{\sqrt{1-t^2}}.$$

(See Freud, §III. 9.) Hence by Lemma 5.1

$$(26) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 f(t) \frac{1}{n \lambda_n(d\alpha, t)} d\alpha(t) = \frac{1}{\pi} \int_{-1}^1 f(t) \frac{dt}{\sqrt{1-t^2}}$$

if  $f$  is continuous on  $[-1, 1]$ . Using one-sided approximation we obtain that (26) remains valid if  $f$  is the characteristic function of a  $d\alpha$  measurable interval  $\Delta \subset (-1, 1)$ . Now we repeat the proof of Theorem 54.

### 6.3. Generalized Christoffel Functions

Definition 1. Let  $0 < p < \infty$ . Then the generalized Christoffel function  $\lambda_n(d\alpha, p, x)$  is defined by

$$\lambda_n(d\alpha, p, x) = \inf_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p d\alpha(t).$$

Later we shall see why we do not introduce a normalization;

$$\lambda_n(d\alpha, p, x) = \inf_{\pi_{n-1}} | \cdot |^{1/p}.$$

Properties 2. (i) If  $d\alpha \leq d\beta$  then  $\lambda_n(d\alpha, p, x) \leq \lambda_n(d\beta, p, x)$ .

(ii)  $\lambda_n(d\alpha, 2, x) \equiv \lambda_n(d\alpha, x)$ . (iii) If  $\text{supp}(d\alpha)$  is compact then

$$\lambda_n(d\alpha, p, x) = \min_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p d\alpha(t).$$

Proof. Let us fix  $\alpha, n, p$  and  $\Delta \supset \Delta(d\alpha)$ . Let us show that  $\lambda_n(d\alpha, p, y) \geq \lambda > 0$  for  $y \in \Delta$ . Let  $m$  be an integer such that  $m \geq p$ . Let  $y \in \Delta \supset \Delta(d\alpha)$ . Then

$$\pi_{n-1}^m(y) = \int_{\pi_{n-1}^m(t)} K_{m,n}(d\alpha, y, t) d\alpha(t)$$

for  $y \in \Delta$ . Hence

$$\max_{y \in \Delta} |\pi_{n-1}(y)|^m \leq C \int_{\Delta} |\pi_{n-1}(t)|^m d\alpha(t)$$

where  $C = C(n, m, d\alpha)$  does not depend on  $\pi_{n-1}$ . Writing  $|\pi_{n-1}(t)|^m = |\pi_{n-1}(t)|^p |\pi_{n-1}(t)|^{m-p}$  ( $m-p \geq 0$ ) we obtain

$$\max_{y \in \Delta} |\pi_{n-1}(y)|^p \leq C \int_{\Delta} |\pi_{n-1}(t)|^p d\alpha(t),$$

in particular  $\lambda_n(d\alpha, p, \gamma) \geq \lambda > 0$  for  $\gamma \in \Delta$ . Thus

$$\int_{-\infty}^{\infty} |\pi_{n-1}(y)| d\alpha(y) \leq \lambda \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^{\frac{1}{p}} d\alpha(t).$$

If we write  $\pi_{n-1}(x) = \sum_{k=0}^{n-1} a_k p_k(d\alpha, x)$  then we obtain from the previous inequality that

$$a_k \leq C \left[ \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p d\alpha(t) \right]^{\frac{1}{p}}$$

where  $C = C(n, p, d\alpha)$  does not depend on  $\pi_{n-1}$ . Now (iii) follows from Bolzano-Weierstrass' theorem by standard arguments.

First of all we shall investigate the simplest case, that is when  $\alpha$  is a Jacobi weight. Let us recall that the Jacobi weight is denoted by  $u = u(a, b)$ . We shall find the exact order of  $\lambda_n(u, p, x)$  on  $[-1, 1]$  when  $n \rightarrow \infty$ .

Definition 3. We write  $\varphi_n(x) \sim \psi_n(x)$  if for every  $n$  and for every  $x$  in consideration (usually for  $-1 \leq x \leq 1$ )

$$0 < c_1 \leq \varphi_n(x)/\psi_n(x) \leq c_2 < \infty.$$

$\varphi(x) \sim \psi(x)$ ,  $n \sim m$  etc. have similar meanings.

Definition 4. Let  $a, b \in \mathbb{R}$ . Then  $u_n$  is defined by

$$u_n(x) \equiv u_n^{(a, b)}(x) = \left[ \sqrt{1-x} + \frac{1}{n} \right]^{2a+1} \left[ \sqrt{1+x} + \frac{1}{n} \right]^{2b+1}$$

for  $x \in [-1, 1]$ .

Let us remark that if  $m \sim n$  then  $u_n(x) \sim u_m(x)$ .

Lemma 5. We have

$$\lambda_n(u^{(a, b)}, x) \sim \frac{1}{n} u_n^{(a, b)}(x)$$

for  $-1 \leq x \leq 1$ .

Proof. See [11].

Lemma 6. Let  $\cos \theta_{kn} = x_{kn}(u)$  for  $k = 0, 1, \dots, n+1$  with  $x_{0n} = 1$  and  $x_{n+1, n} = -1$ . Then

$$\theta_{kn} - \theta_{k-1, n} \sim \frac{1}{n}$$

for  $k = 1, 2, \dots, n+1$ .

Proof. See e.g. [12].

Corollary 7. Let  $x \in [x_{kn}(u), x_{k-1, n}(u)]$  ( $k = 1, 2, \dots, n+1$ ). Then

$$u_n^{(a_1, b_1)}(x) \sim u_n^{(a_1, b_1)}(x_k) \sim u_n^{(a_1, b_1)}(x_{k-1})$$

for  $a_1, b_1 \in \mathbb{R}$ .

Let  $v$  be as before - the Tschebyshev weight. Then

$$|K_n(v, x, t)| \leq C \min\left\{n, \frac{1}{|x-t|}\right\}$$

for  $x, t \in [-1, 1]$ . This estimate is good inside  $(-1, 1)$  but not for  $x$  and  $t$  close to the endpoints of  $[-1, 1]$ . We shall need a better estimate which we formulate as

Lemma 8. Let  $x, t \in [-1, 1]$ . Then

$$|K_n(v, x, t)| \leq C \min \left\{ \frac{\sqrt{1-x^2} + \sqrt{1-t^2}}{|x-t|} \right\}.$$

Proof. The idea comes from Pollard probably:

$$\begin{aligned} T_n(x) T_{n-1}(t) - T_{n-1}(x) T_n(t) &= [T_n(x) - T_{n-1}(x)] T_{n-1}(t) + \\ &+ T_{n-1}(x) [T_n(t) - T_{n-1}(t)]. \end{aligned}$$

Lemma 9. Let  $0 < \delta < 1$  and  $a, b, c \in \mathbb{R}$ . Then we have uniformly in  $n$  and  $m$  such that  $1 \leq m \leq \delta n$

$$\begin{aligned} \sum_{k=1}^n k^a |k+m| b |k-m|^c &\sim \\ \sim m^{b+c} \begin{cases} 1 & \text{if } a < -1 \\ \log(m+2) & \text{if } a = -1 \\ m^{a+1} & \text{if } a > -1 \end{cases} + \\ + m^{a+b} \begin{cases} 1 & \text{if } c < -1 \\ \log(m+2) & \text{if } c = -1 \\ m^{c+1} & \text{if } c > -1 \end{cases} + \\ + \begin{cases} m^{a+b+c+1} & \text{if } a+b+c < -1 \\ \log \frac{n}{m} + m^{a+b+c+1} & \text{if } a+b+c = -1 \\ n^{a+b+c+1} & \text{if } a+b+c > -1 \end{cases}. \end{aligned}$$

Proof. We write

$$\sum_{k=1}^n \left( \frac{m}{2} \right) + \sum_{k=m+1}^{m-1} \left( \frac{m}{2} (\delta^{-1} + 1) \right) + \sum_{k=\lfloor \frac{m}{2} (\delta^{-1} + 1) \rfloor + 1}^n \left( \frac{m}{2} (\delta^{-1} + 1) \right)$$

and each sum can easily be estimated.

Now we can compute the following important

Theorem 10. Let  $u = u^{(a, b)}$  be given. Then there exists a natural integer  $N_1 = N_1(a, b)$  such that for every fixed  $N > N_1$

$$(1) \quad \int_{-1}^1 \frac{K_n(v, x, t)}{K_n(v, x, x)} \left| \frac{K_n(v, x, t)}{K_n(v, x, x)} \right|^N u(t) dt \leq C \frac{1}{n} u_n^{(a, b)}(x)$$

for  $-1 \leq x \leq 1$  and  $n = 1, 2, \dots$

Proof. Let us note that if (1) holds for  $N_1 = N$  then it holds also for every fixed  $N > N_1$ . Let  $N$  be an even natural integer. Then  $n \sim nN \equiv M$  and by Definition 4  $u_n(x) \sim u_M(x)$  for  $|x| \leq 1$ . Let us compute the integral on the left side of (1) by the Gauss-Jacobi mechanical quadrature formula. We can do this since  $N$  is even. The above integral equals to

$$(2) \quad \sum_{k=1}^M \lambda_{kM}(u) \left[ \frac{K_n(v, x, x_{kM}(u))}{K_n(v, x, x)} \right]^N$$

and we shall estimate this sum. We may suppose without loss of generality that  $0 \leq x \leq 1$ . Take a suitable value for  $N$  such that (1) holds. If  $-1 \leq x < 0$  then by similar arguments we obtain another value for  $N$  such that (1) is satisfied. Taking the maximum of these two values of  $N$  we

get a new  $N$  which is good for every  $x \in [-1, 1]$ . We have  
 $K_n(v, x, x) \sim M^{-N}$  for  $-1 \leq x \leq 1$  by Example 6.2.8. Let  $m$  be defined  
 by

$$|x - x_{kM}| \geq |x - x_{mM}| \text{ for } k = 1, 2, \dots, M.$$

Then

$$(2) \sim M^{-N} \left\{ \sum_{x_k \leq -\frac{1}{2}} + \sum_{-\frac{1}{2} < x_k < 1} + \sum_{k=m}^M \right\} \lambda_{kM}(u) K_n^N(v, x, x_{kM}).$$

By Lemma 8 the first sum here is  $O(1)$ . By Lemma 5 and Corollary 7 the  
 last sum is of order  $M^{N-1} u_M(x)$ . Hence

$$(2) \leq C[M^{-N} + \frac{1}{M} u_M(x)] +$$

$$+ M^{-N} \sum_{-\frac{1}{2} < x_k < 1} \lambda_{kM}(u) K_n^N(v, x, x_{kM}).$$

By Lemma 6 for  $k \neq m$   $|x - x_{kM}| > C \frac{1}{M} \sqrt{1-x^2}$  and  $|x - x_{kM}| \geq$   
 $\geq C \frac{1}{M^2} |k-m| |k+m|$ . Consequently by Lemmas 5, 6 and 8

$$M^{-N} \sum_{-\frac{1}{2} < x_k < 1} \lambda_{kM}(u) K_n^N(v, x, x_{kM}) \leq$$

$$\leq CM^{-N-1} \sum_{-\frac{1}{2} < x_k < 1} (1 - x_{kM})^{a+\frac{1}{2}} \left( \frac{\sqrt{1-x^2} + \sqrt{1-x_{kM}^2}}{x - x_{kM}} \right)^N \leq$$

-145-

$$\leq CM^{-N-1} (1-x)^{\frac{2}{2}} \sum_{k \neq m}^N \frac{1}{(\frac{k}{M})^{2a+1}} \left[ \frac{(k-m)(k+m)}{M^2} \right]^{-N} +$$

$$+ CM^{-N-1} \sum_{k \neq m}^N \frac{1}{(\frac{k}{M})^{2a+1+N}} \left[ \frac{(k-m)(k+m)}{M^2} \right]^{-N} \equiv A+B$$

where  $0 < c_1 < 1$ . To estimate  $A$  and  $B$  we use Lemma 9. We obtain  
 that if

$$N > 1 \text{ and } 2a+1-2N < -1$$

then

$$A \sim (1-x)^{\frac{2}{2}} M^{-N-2a-2a+1-N} \sim (1-x)^{\frac{2}{2}} \frac{1}{M} \left[ \sqrt{1-x^2} + \frac{1}{M} \right]^{2a+1-N} \leq$$

$$\leq C \frac{1}{M} u_M^{(a,b)}(x),$$

and if

$$1, > 1 \text{ and } 2a+1-N < -1$$

then

$$B \sim M^{-2-2a-2a+1} \frac{1}{M} u_M^{(a,b)}(x).$$

Thus for  $N > \max\{1, 2a+2\}$

$$(2) \leq C[M^{-N} + \frac{1}{M} u_M^{(a,b)}(x)] \leq C \frac{1}{M} u_M^{(a,b)}(x) \leq$$

$$\leq C \frac{1}{n} u_n^{(a,b)}(x).$$

-146-



To finish the proof of the theorem we choose  $N > 0$  that  $N > \max(1, 2a+2, 2b+2)$ .

Lemma 11. Let  $a \geq -\frac{1}{2}$ ,  $0 < p < \infty$ . Then

$$(3) \quad |\pi_{n-1}(x)|^p \leq C n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

for  $-\frac{1}{2} \leq x \leq 1$  and there exists a number  $c_1 = c_1(a, p) > 0$  such that

$$(4) \quad \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt \leq 2 \int_0^{1-c_1 n^{-2}} |\pi_{n-1}(t)|^p (1-t)^a dt.$$

Proof. Let  $k$  be a natural integer such that  $2k \geq p$ . Then  $\deg \pi_{n-1}^k \sim n$  and by Lemma 5

$$\pi_{n-1}^{2k}(x) \leq C n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^{2k} (1-t)^a dt$$

for  $-1 \leq x \leq 1$ , that is

$$\max_{|x| \leq 1} |\pi_{n-1}(x)|^{2k} \leq \max_{|x| \leq 1} |\pi_{n-1}(x)|^{2k-p} \cdot$$

$$C n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t)^a dt.$$

Therefore

$$|\pi_{n-1}(x)|^p \leq C n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

for  $-1 \leq x \leq 1$ . Let us put here  $x^M \pi_{n-1}^2(x^2)$  instead of  $\pi_{n-1}(x)$ . Then for  $M$  fixed

$$|x^{2M} \pi_{n-1}(x^2)|^p \leq C n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p t^{Mp-\frac{1}{2}} (1-t)^a dt$$

for  $0 \leq x \leq 1$ . Let now  $M$  be so large that  $Mp - \frac{1}{2} \geq 0$ . Hence (3) holds. We obtain from (3) that for  $c_1 > 0$

$$\int_{1-\frac{1}{n^2}}^1 |\pi_{n-1}(x)|^p (1-x)^a dx \leq C n^{2(a+1)} \int_{1-\frac{1}{n^2}}^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

$$\cdot \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

and if  $c_1 > 0$  is small then

$$(5) \quad C n^{2(a+1)} \int_{1-\frac{1}{n^2}}^1 |\pi_{n-1}(t)|^p (1-t)^a dt \leq \frac{1}{2},$$

which implies (4).

Lemma 12. Let  $\min(a, b) \geq -\frac{1}{2}$ ,  $0 < p < \infty$ . Then

$$(6) \quad \lambda_n(u, x) \leq C \lambda_n(u, p, x)$$

for  $n = 1, 2, \dots$  and  $-1 \leq x \leq 1$ .

Proof. Let  $k$  be an integer such that  $2k \geq p$ . Put  $a_1 = (a + \frac{1}{2}) \frac{1}{p} - \frac{1}{2}$  and  $b_1 = (b + \frac{1}{2}) \frac{2k}{p} - \frac{1}{2}$ . Then  $a_1, b_1 \geq -\frac{1}{2}$  and by Lemma 5 and 11 there exists  $\varepsilon > 0$  such that

$$\begin{aligned} u^{(a_1, b_1)}(x) \sqrt{1-x^2} \pi_{n-1}^{2k}(x) &\leq \\ &\leq C n \int_{-1+\frac{\varepsilon}{n^2}}^{1-\frac{\varepsilon}{n^2}} \pi_{n-1}^{2k}(t) u^{(a_1, b_1)}(t) \sqrt{1-t^2} dt \end{aligned}$$

for  $|x| \leq 1 - \frac{\varepsilon}{n}$ . Hence

$$\begin{aligned} & |[(u^{(a)}, b)](x) \sqrt{1-x}^{\frac{1}{2k}} \pi_{n-1}(x)|^p \leq \\ & \leq C n \int_n^{1-\frac{\varepsilon}{n}} |(u^{(a)}, b)](t) \sqrt{1-t}^{\frac{1}{2k}} \pi_{n-1}(t)|^p \frac{1}{\sqrt{1-t}} dt, \\ & \quad -1 + \frac{\varepsilon}{2n} \end{aligned}$$

that is

$$|\pi_{n-1}(x)|^p \leq n u_n^{(a,b)}(x)^{-1} \int_{-1}^1 |\pi_{n-1}(t)|^p u(t) dt$$

for  $|x| \leq 1 - \frac{\varepsilon}{n}$ . By Lemma 11 the latter inequality also holds for

$$1 - \frac{\varepsilon}{2n} \leq |x| \leq 1. \text{ Now (6) follows from Lemma 5.}$$

Theorem 13. Let  $u$  be a Jacobi weight and  $0 < p < \infty$ . Then

$$\lambda_n(u, p, x) \sim \lambda_n(u, x)$$

for  $-1 \leq x \leq 1$ .

Proof. Let  $N$  be such that (1) holds. Let  $m \in \mathbb{N}$  be so big that  $mp > N$ . If we put  $n_1 = \lfloor \frac{n}{m} \rfloor$  then

$$\lambda_n(u, p, x) \leq \int_{-1}^1 \left| \frac{K_{n_1}(v, x, t)}{K(v, x, x)} \right|^{mp} u(t) dt \leq C \frac{1}{n_1} u_{n_1}(x) \sim$$

$$\sim \frac{1}{n} u_n(x) \sim \lambda_n(u, x) \quad (|x| \leq 1)$$

by Lemma 5 and Theorem 10. Now we shall show that

$$(7) \quad \lambda_n(u, x) \leq C \lambda_n(u, p, x)$$

for  $|x| \leq 1$ . Let  $u = u^{(a,b)}$ ,  $u^* = u^{(\max\{a, -\frac{1}{2}\}, \max\{b, -\frac{1}{2}\})}$  and

$$\hat{u} = u^{(\max\{-(\frac{a+1}{2}), -\frac{1}{2}\}, \max\{-(\frac{b+1}{2}), -\frac{1}{2}\})}.$$

We have by Lemma 5

and 12

$$|\pi_{n-1}(x) K_n(\hat{u}, x, x)|^p \leq C n |u_n^*(x)|^{-1}.$$

$$\cdot \int_{-1}^1 |\pi_{n-1}(t) K_n(\hat{u}, t, t)|^p u^*(t) dt \quad (|x| \leq 1),$$

$$|K_n(\hat{u}, t, t)|^p u^*(t) \leq C n^p u(t) \quad (|t| \leq 1)$$

and

$$u_n^*(x) |K_n(\hat{u}, x, x)|^p \sim n^p u_n(x)$$

for  $|x| \leq 1$ . Hence

$$|\pi_{n-1}(x)|^p \leq u_n(x)^{-1} \int_{-1}^1 |\pi_{n-1}(t)|^p u(t) dt.$$

From this inequality and Lemma 5 we obtain (7).

There is a very important consequence of Theorem 13 which we

formulate as

Theorem 14. Let  $u$  be a Jacobi weight and  $0 < p < \infty$ . Then there exists

a number  $C_1 = C_1(u, p) > 0$  such that

$$\int_{-1}^1 |\pi_n(t)|^p u(t) dt \leq 2 \int_{-1+\frac{1}{n}}^{\frac{C_1}{1-\frac{1}{n}}} |\pi_n(t)|^p u(t) dt$$

for every  $\pi_n$ .

Proof. Use Lemma 5, Theorem 13 and (5).

Corollary 15. Let  $u$  be a Jacobi weight,  $0 < p < \infty$ ,  $\epsilon > 0$ . Then for every  $\pi_n$

$$\int_{-1}^1 |\pi_n(t)|^p u(t) dt \leq C n^{2\epsilon} \int_{-1}^1 |\pi_n(t)|^p u(t) (1-t^2)^{\epsilon} dt$$

with  $C = C(p, u, \epsilon)$ .

Corollary 16. Let  $0 < q < p < \infty$ . Let  $u^p$  be a Jacobi weight. Then

$$\left( \int_{-1}^1 |\pi_n(t) u(t)|^p dt \right)^{\frac{1}{p}} \leq C n^{\frac{2}{q}(\frac{1}{p} - \frac{1}{q})} \left( \int_{-1}^1 |\pi_n(t) u(t)|^q dt \right)^{\frac{1}{q}}$$

for every  $\pi_n$  where  $C = C(p, q, u)$ .

Proof. By Lemma 5 and Theorems 13, 14

$$\begin{aligned} \int_{-1}^1 |\pi_n(t) u(t)|^p dt &\leq 2 \int_{-1+\frac{1}{n}}^{1-\frac{1}{n}} |\pi_n(t) u(t)|^{p-q+q} dt \leq \\ &\leq C n^{\frac{2-p-q}{p}} \left( \int_{-1+\frac{1}{n}}^{1-\frac{1}{n}} |\pi_n(t) u(t)|^p dt \right)^{\frac{p-q}{p}} \int_{-1+\frac{1}{n}}^{1-\frac{1}{n}} |\pi_n(t) u(t)|^q dt. \end{aligned}$$

Let us note that Corollaries 15 and 16 for  $1 \leq p < \infty$  and  $1 \leq q < p < \infty$  are not new. (See Khalilova [9].)

Before we begin to investigate the generalized Christoffel functions for weights different from the Jacobi ones we shall need a few lemmas.

Lemma 17. Let  $\alpha(x) + \alpha(-x) \equiv \text{const}$ . Let  $\alpha_1$  and  $\alpha_2$  be defined by

-151-

$$\alpha_1(x) = \begin{cases} 0 & \text{for } x < 0 \\ \alpha(\sqrt{x}) - \alpha(0) & \text{for } x \geq 0 \end{cases}$$

and

$$\alpha_2(x) = \begin{cases} 0 & \text{for } x < 0 \\ \int_0^x t d\alpha(\sqrt{t}) & \text{for } x \geq 0. \end{cases}$$

Then

$$\sum_{k=0}^n p_k^2(d\alpha, x) = \begin{cases} \frac{1}{2} \lambda_{\frac{n}{2}+1}^{-1}(d\alpha_1, x^2) & \text{for } n \text{ even} \\ \frac{x}{2} \lambda_{\frac{n+1}{2}}^{-1}(d\alpha_2, x^2) & \text{for } n \text{ odd} \end{cases}$$

and

$$\sum_{\substack{k=0 \\ k \equiv n \pmod{2}}}^n p_k^2(d\alpha, x) = \max_{\pi_n \equiv n \pmod{2}} \frac{\pi_n^2(x)}{\int_{-\infty}^{\infty} \pi_n^2(t) d\alpha(t)} = 1,$$

where  $\pi_n \equiv n \pmod{2}$  means that  $\pi_n$  is even if  $n$  is even and  $\pi_n$  is odd if  $n$  is odd.

Proof. Easy calculation.

Having in mind later applications we shall prove the following

Lemma 18. Let  $\alpha \in M(0, 1)$  and let  $\alpha(x) + \alpha(-x) \equiv \text{const}$ . Let  $\alpha_1$  and  $\alpha_2$  be defined as in Lemma 17. Then  $\alpha_1, \alpha_2 \in M(\frac{1}{2}, \frac{1}{2})$ .

-152-

Proof. The lemma follows from the relations

$$\alpha_n(d\alpha_1) = \frac{Y_{2n}^2(d\alpha)}{Y_{2n+1}^2(d\alpha)} + \frac{Y_{2n-1}^2(d\alpha)}{Y_{2n}^2(d\alpha)},$$

$$\frac{Y_{n-1}(d\alpha_1)}{Y_n(d\alpha_1)} = \frac{Y_{2n-2}(d\alpha)}{Y_{2n}(d\alpha)}$$

$$\alpha_n(d\alpha_2) = \frac{Y_{2n+1}^2(d\alpha)}{Y_{2n+2}^2(d\alpha)} + \frac{Y_{2n}^2(d\alpha)}{Y_{2n+1}^2(d\alpha)},$$

$$\frac{Y_{n-1}(d\alpha_2)}{Y_n(d\alpha_2)} = \frac{Y_{2n-1}(d\alpha)}{Y_{2n+1}(d\alpha)}$$

which can easily be checked.

Lemma 19. Let  $w(t) = |t|^\Gamma (1-t)^{2a}$  for  $-1 \leq t \leq 1$  with  $\Gamma, a > -1$  and  $\text{supp}(w) = [-1, 1]$ . Then

$$\lambda_n(w, x) \sim \frac{1}{n} \left( |x| + \frac{1}{n} \right)^\Gamma \left( \sqrt{1-x} + \frac{1}{n} \right)^{2a+1} \sqrt{1+x + \frac{1}{n}}^{2a+1}$$

for  $-1 \leq x \leq 1$ .

Proof. Apply Lemmas 5 and 17.

Theorem 20. Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $t \in \Delta^0$ ,  $\Gamma > -1$ .

Let  $\alpha$  be absolutely continuous in  $\Delta$  and let

$$\alpha'(t) \sim |t-t^*|^\Gamma \quad (t \in \Delta).$$

Then

$$\lambda_n(d\alpha, x) \sim \frac{1}{n} \left( |x-t^*| + \frac{1}{n} \right)^\Gamma$$

for  $x \in \Delta_1 \subset \Delta^0$ .

Proof. By Lemma 19 we have to show that

$$(8) \quad \lambda_n(d\alpha, x) \leq C \frac{1}{n} \left( |x-t^*| + \frac{1}{n} \right)^\Gamma$$

for  $x \in \Delta_1 \subset \Delta^0$ . Let  $\hat{v}$  denote the Tschebyshev weight corresponding to  $\Delta(d\alpha)$  and  $m$  be a natural integer. Then

$$\lambda_n(d\alpha, x) \leq \int_{-\infty}^{\infty} \left[ \frac{K\left(\frac{n}{m}\right) \left(\hat{v}, x, t\right)}{K\left(\frac{n}{m}\right) \left(\hat{v}, x, x\right)} \right]^{2m} d\alpha(t)$$

and hence

$$\lambda_n(d\alpha, x) \leq C n^{-2m} \int_{\Delta} d\alpha(t) + C \int_{t \in \Delta} |t-t^*|^\Gamma \frac{1}{1+n} \frac{1}{(x-t)^{2m}} dt$$

uniformly for  $x \in \Delta_1 \subset \Delta^0$  if  $m$  is fixed. The second integral on the right side of the latter inequality can be estimated by standard methods.

Finally we obtain that (8) is satisfied if we choose  $m$  large enough.

Let us note that the calculation in Theorem 20 was simple because we needed estimates only for  $x \in \Delta_1 \subset \Delta^0$  and not for  $x \in \Delta$ .

Lemma 21. Let  $0 < p < \infty$ ,  $\Gamma \geq 0$ ,  $0 \in \Delta_1^0$ ,  $\Delta_1 \subset \Delta^0$ . Then for every  $n, l$

$$|\pi_{n-l}(x)|^p \leq C n^{\Gamma+1} \int_{\Delta} |\pi_{n-l}(t)|^p |t|^\Gamma dt$$

uniformly for  $x \in \Delta_1$ .



Lemma 23. Let  $0 < p < \infty$ ,  $\Gamma \geq 0$ ,  $0 \in \Delta_1^0$ ,  $\Delta_1 \subset \Delta^0$ . Let  $w(t) = |t|^\Gamma$  for  $t \in \Delta$  with  $\text{supp}(w) = \Delta$ . Then

$$(9) \quad \frac{1}{n} \left( |x| + \frac{1}{n} \right)^\Gamma \leq C \lambda_n(w, p, x) \quad (n = 1, 2, \dots)$$

uniformly for  $x \in \Delta_1$ .

Proof. Let  $\varepsilon > 0$  be such that  $[-2\varepsilon, 2\varepsilon] \subset \Delta_1^0$ . If  $x \in \Delta_1 \setminus [-\varepsilon, \varepsilon]$  then (9) follows from Lemma 5 and Theorem 13. If  $x \in [-\varepsilon, \varepsilon]$  then we

find an integer  $m$  such that  $2m \geq p$ . We put  $w^*(t) = |t|^{\frac{2m\Gamma}{p} - \frac{1}{2}}$  for  $|t| \leq 2\varepsilon$  with  $\text{supp}(w^*) = [-2\varepsilon, 2\varepsilon]$ . Then by Lemmas 19 and 22

$$\sqrt{4\varepsilon^2 - x^2} w^*(x) |\pi_{n-1}(x)|^{2m} \leq C n \int_{\frac{C}{n}}^{\frac{1}{n}} |\pi_{n-1}(t)|^{2m} w^*(t) dt$$

for  $\frac{C}{n} \leq |x| \leq 2\varepsilon$ . Hence

$$\sqrt{4\varepsilon^2 - x^2} w(x) |\pi_{n-1}(x)|^p \leq C n \int_{\frac{C}{n}}^{\frac{1}{n}} |\pi_{n-1}(t)|^p w(t) dt \leq \frac{C}{n} \leq |t| \leq 2\varepsilon$$

$$\leq C n \int_{\Delta} |\pi_{n-1}(t)|^p w(t) dt$$

for  $\frac{C}{n} \leq |x| \leq 2\varepsilon$ . Hence (9) holds also for  $\frac{C}{n} \leq |x| \leq \varepsilon$ . If  $|x| \leq \frac{C}{n}$  we apply Lemma 21.

Lemma 24. Lemma 23 remains true if  $-1 < \Gamma < 0$  instead of  $\Gamma \geq 0$ .

Proof. Let  $\varepsilon > 0$  be such that  $\Delta \subset (-\varepsilon, \varepsilon)$ . Let  $w^*(t) = |t|^{-\frac{1}{p}}$  for  $|t| \leq \varepsilon$  with  $\text{supp}(w^*) = [-\varepsilon, \varepsilon]$ . Then by Lemma 19

Proof. Let  $\varepsilon > 0$  be such that  $[-2\varepsilon, 2\varepsilon] \subset \Delta_1^0$ . If  $x \in \Delta_1 \setminus [-\varepsilon, \varepsilon]$  then by Lemma 5 and Theorem 13

$$|\pi_{n-1}(x)|^p \leq C n \int_{\Delta \setminus [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]} |\pi_{n-1}(t)|^p dt \leq$$

$$\leq C_1 n^{\Gamma+1} \int_{\Delta} |\pi_{n-1}(t)|^p |t|^\Gamma dt$$

since  $\Gamma \geq 0$ . Let now  $x \in [-\varepsilon, \varepsilon]$ . Let  $v_\varepsilon$  denote the Tschebyshev weight corresponding to  $[-2\varepsilon, 2\varepsilon]$ ,  $M$  be an even natural integer and  $m$  be an integer such that  $2m \geq p$ . Then by Lemma 19

$$[v_\varepsilon(x)]^{-M} |\pi_{n-1}(x)|^{2m} v_\varepsilon(x)^{-1} \leq C n^{\Gamma+1} \int_{-2\varepsilon}^{2\varepsilon} [v_\varepsilon(t)]^{-M} |\pi_{n-1}(t)|^{2m} |t|^\Gamma dt$$

for  $x \in [-2\varepsilon, 2\varepsilon]$ . Hence

$$\left| v_\varepsilon(x) \frac{1}{2m} \pi_{n-1}(x) \right|^p \leq C n^{\Gamma+1} \int_{-2\varepsilon}^{2\varepsilon} |\pi_{n-1}(t)|^p v_\varepsilon(t)^{-Mp - \frac{p}{2m} + 1} dt$$

$$= |t|^\Gamma dt$$

for  $x \in [-2\varepsilon, 2\varepsilon]$ . Let now  $M$  be so large that  $-Mp - \frac{p}{2m} + 1 \leq 0$ .

From Lemma 21 we obtain the following

Lemma 22. Let  $0 < p < \infty$ ,  $\Gamma \geq 0$ ,  $0 \in \Delta^0$ . Then there exists

$C_1 = C_1(p, \Gamma, \Delta)$  such that

$$\int_{\Delta} |\pi_{n-1}(t)|^p |t|^\Gamma dt \leq 2 \int_{\frac{C_1}{n}}^{\varepsilon} |\pi_{n-1}(t)|^p |t|^\Gamma dt$$

$$\lambda_n(w^*, x) \sim \frac{1}{n}(|x| + \frac{1}{n})^{-\frac{\Gamma}{p}}$$

for  $x \in \Delta$ . By Lemma 23

$$|\pi_{n-1}(x) \lambda_n^{-1}(w^*, x)|^p \leq C n \int_{\Delta} |\pi_{n-1}(t) \lambda_n^{-1}(w^*, t)|^p dt$$

for  $x \in \Delta_1$ . Hence

$$|\pi_{n-1}(x)|^p (|x| + \frac{1}{n})^\Gamma \leq C n \int_{\Delta} |\pi_{n-1}(t)|^p |t|^\Gamma dt$$

for  $x \in \Delta_1$ .

Theorem 25. Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $t \in \Delta^0$ ,  $\Gamma > -1$ ,  $0 < p < \infty$ . Let  $\alpha$  be absolutely continuous in  $\Delta$  and let

$$\alpha'(t) \sim |t - t^*|^\Gamma \quad (t \in \Delta).$$

Then

$$\lambda_n(d\alpha, p, x) \sim \lambda_n(d\alpha, x) \sim \frac{1}{n} (|x - t^*| + \frac{1}{n})^\Gamma$$

for  $x \in \Delta_1 \subset \Delta^0$ .

Proof. The inequality

$$\lambda_n(d\alpha, p, x) \leq \frac{C}{n} (|x - t^*| + \frac{1}{n})^\Gamma \quad (x \in \Delta_1 \subset \Delta^0)$$

can be proved exactly by the same way as in Theorem 20 for  $p = 2$ . For the estimate from below we can suppose that  $t^* \in \Delta_1^0$  and then we apply

Lemmas 23 and 24.

Corollary 26. Lemma 22 remains valid for  $-1 < \Gamma < 0$  and consequently

if  $\Gamma > -1$ ,  $\varepsilon > 0$ ,  $0 \in \Delta^0$  and  $0 < p < \infty$  then for every  $\pi_n$

$$\int_{\Delta} |\pi_n(t)|^p |t|^\Gamma dt \leq C n^\varepsilon \int_{\Delta} |\pi_n(t)|^p |t|^{\Gamma+\varepsilon} dt$$

where  $C = C(p, \Gamma, \varepsilon, \Delta)$ .

Theorem 27. Let  $\text{supp}(d\alpha)$  be compact,  $0 < p < \infty$ ,  $a > -1$ . Let

$\Delta(d\alpha) = [c_1, c_2]$ ,  $\delta > 0$  and let  $\alpha$  be absolutely continuous in  $[c_2 - \delta, c_2]$ .

Let

$$\alpha'(t) \sim (c_2 - t)^a \quad (t \in [c_2 - \delta, c_2]).$$

Then

$$\lambda_n(d\alpha, p, x) \sim \frac{1}{n} (\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for  $x \in [c_2 - \frac{\delta}{2}, c_2]$ .

Proof. We have by Theorem 10 and standard arguments

$$\lambda_n(d\alpha, p, x) \leq \frac{1}{n} (\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for  $x \in [c_2 - \frac{\delta}{2}, c_2]$ . The converse inequality follows from Lemma 5 and Theorem 13.

From Theorems 25 and 27 we obtain

Theorem 28. Let  $1 = t_1 > t_2 > \dots > t_N = -1$ ,  $\Gamma_k > -1$  for  $k = 1, 2, \dots, N$  and let  $w$  be defined by

$$w(t) = \prod_{k=1}^N |t - t_k|^{\Gamma_k} \quad (-1 \leq t \leq 1)$$

with  $\text{supp}(w) = [-1, 1]$ . Let

$$\bar{w}_n(t) = \left( \sqrt{1-t} + \frac{1}{n} \right)^{2\Gamma+1} \prod_{k=2}^{N-1} \left( 1-t_k + \frac{1}{n} \right)^{\Gamma} k_{\left( \sqrt{1-t} + \frac{1}{n} \right)}^{2\Gamma+1} N^{+1}$$

for  $-1 \leq t \leq 1$ . Then for every  $0 < p < \infty$

$$\lambda_n(w, p, x) \sim \lambda_n(w, x) \sim \frac{1}{n} \bar{w}_n(x)$$

for  $|x| \leq 1$ .

Corollary 29. We can establish inequalities similar to those in Theorem 14 and Corollaries 15 and 16.

The exact formulation of Corollary 29 is left to the reader.

Lemma 30. Let  $p > 1$ . Then

$$\int_{-1}^1 |K_n(v, x, t)|^p v(t) dt \sim n^{p-1} \sim \lambda_n(v, x)^{1-p}$$

for  $-1 \leq x \leq 1$ .

Proof. The estimate

$$n^{p-1} \leq C \int_{-1}^1 |K_n(v, x, t)|^p v(t) dt$$

follows immediately from Theorem 13. The converse estimate is obvious when  $p \geq 2$  and can be obtained by a simple calculation from Lemma 8 when  $1 < p < 2$ .

Lemma 31. Let  $\alpha$  be an arbitrary weight,  $p > 1$ . Then for almost every  $x \in [-1, 1]$

$$\lim_{n \rightarrow \infty} \frac{\int_{-1}^1 |K_n(v, x, t)|^p d\alpha(t)}{\int_{-1}^1 |K_n(v, x, t)|^p v(t) dt} = \sqrt{1-x^2} \alpha'(x).$$

Proof. Using Lemma 30 the lemma can be proved in almost the same way as Lemma 6.2.32. We shall not go into details.

Theorem 32. Let  $\text{supp}(d\alpha) \subset [-1, 1]$  and  $0 < p < \infty$ . Then

$$(10) \quad \limsup_{n \rightarrow \infty} \lambda_n(d\alpha, p, x) \leq C \alpha'(x) \sqrt{1-x^2}$$

for almost every  $x \in [-1, 1]$  where  $C = C(p)$ .

Proof. Let  $m$  be a natural integer such that  $mp > 1$ . Then

$$\lambda_n(d\alpha, p, x) \leq \int_{-1}^1 \left| \frac{K_{\left[ \frac{n}{m} \right]}(v, x, t)}{K_{\left[ \frac{n}{m} \right]}(v, x, x)} \right|^{mp} d\alpha(t).$$

Now we apply Lemmas 30 and 31.

Theorem 33. Let  $\alpha \in M(0, 1)$  and  $0 < p < \infty$ . Then for almost every  $x \in \text{supp}(d\alpha)$  (10) holds with  $C = C(p)$ .

Proof. Combine the arguments used in the proof of Theorems 32 and 6.2.51.

Theorem 34. Let  $\alpha$  be an arbitrary weight. Let  $\Delta$  and  $\varepsilon > 0$  be given and let  $v_\Delta$  denote the Tschebyshev weight corresponding to  $\Delta$ . Let  $\{\alpha'\}^\varepsilon \in L^1(\Delta)$ . Then for each  $p \in (0, \infty)$

$$\liminf_{n \rightarrow \infty} n \lambda_n(d\alpha, p, x) \geq C \alpha'(x) v(x)^{-1}$$

for almost every  $x \in \Delta$  where  $C = C(\epsilon, \Delta, p)$ .

Proof. Let  $q = \epsilon p(1+\epsilon)^{-1}$ ,  $m$  and  $M$  be natural integers such that  $mp\epsilon > 1$  and  $2\epsilon pM > 1+\epsilon$ . Let  $N = \lfloor \frac{n}{m} \rfloor$ . We can suppose without loss of generality that  $\Delta = [-1, 1]$ . Then by Theorem 13

$$|\lambda_N^m(v, x) v(x)^{-2M} \pi_{N-1}(x)|^q \leq C n \int_{-1}^1 |K_N^m(v, x, t) v(t)|^{-2M} \cdot$$

$$\pi_{N-1}(t)|^q v(t) dt.$$

Hence by Hölder's inequality

$$|\pi_{N-1}(x)|^p \leq C n^{\frac{p}{q}} \lambda_N^m(v, x)^{pm} v(x)^{2Mp} \cdot \int_{-1}^1 |\pi_{N-1}(t)|^p \alpha'(t) dt \cdot$$

$$\cdot \left\{ \int_{-1}^1 |K_N^m(v, x, t)|^{\frac{mpq}{p-q}} v(t)^{\frac{p}{p-q}} \pi_{N-1}(t)^{-\frac{q}{p-q}} dt \right\}^{\frac{p-q}{q}}$$

Using Lemma 30 we obtain

$$|\pi_{N-1}(x)|^p \leq C n v(x)^{2Mp} \int_{-\infty}^{\infty} |\pi_{N-1}(t)|^p d\alpha(t) \cdot$$

$$\left\{ \frac{\int_{-1}^1 |K_N^m(v, x, t)|^{mp\epsilon} v(t)^{1+\epsilon-2M\epsilon p} \alpha'(t)^{-\epsilon} dt}{\int_{-1}^1 |K_N^m(v, x, t)|^{m\epsilon p} v(t) dt} \right\}^{1/\epsilon}$$

Consequently

-161-

$$\frac{1}{n \lambda_n(d\alpha, p, x)} \leq C n v(x)^{2Mp} \cdot$$

$$\left\{ \frac{\int_{-1}^1 |K_N^m(v, x, t)|^{m\epsilon p} v(t)^{1+\epsilon-2M\epsilon p} \alpha'(t)^{-\epsilon} dt}{\int_{-1}^1 |K_N^m(v, x, t)|^{m\epsilon p} v(t) dt} \right\}^{1/\epsilon}$$

By the conditions  $v^{1+\epsilon-2M\epsilon p} \alpha'^{-\epsilon}$  is a weight. Thus the theorem follows from Lemma 31.

In Theorem 34 the most important case is when  $p = 2$ . Let us formulate it separately as

Theorem 35. Let  $\epsilon > 0$ . If  $(\alpha')^{-\epsilon} \in L^1(\Delta)$  then

$$\limsup_{n \rightarrow \infty} \frac{\alpha'(x) v(x)^{-1}}{n \lambda_n(d\alpha, x)} \in L^1(\Delta).$$

Let us note that Corollary 6.2.24 is contained in Theorem 35. Hence Theorems 6.2.25 and 6.2.26 remain valid if  $[\alpha']^{-\epsilon} \in L^1(\Delta)$  instead of  $1/\alpha' \in L^1(\Delta)$ .

-162-



7. The Coefficients in the Recurrence Formula.

Theorem 1. Let  $\text{supp } (d\alpha) \subset [-1, 1]$  and

$$\sum_{j=1}^{\infty} \left| \frac{y_{j-1}(d\alpha)}{y_j(d\alpha)} - \frac{1}{2} \right| < \infty.$$

Then  $\alpha \in S$ .

Proof. Let  $0 \leq k \leq n$ . Let us divide both sides in 3.1(2) by  $x^n$  and let

$x \rightarrow \infty$ . We obtain

$$2^{-n} \gamma_n(d\alpha) = 2^{-k} \gamma_k(d\alpha) + \sum_{j=k+1}^n 2^{-j} \gamma_j(d\alpha) \left[ 1 - 2 \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} \right].$$

Let us fix  $k$  so that

$$\sum_{j=k+1}^{\infty} \left| 1 - 2 \frac{\gamma_{j-1}}{\gamma_j} \right| < \frac{1}{2}.$$

Then for every  $n \geq k+1$

$$2^{-n} \gamma_n \leq 2^{-k} \gamma_k + \frac{1}{2} \max_{k+1 \leq j \leq n} 2^{-j} \gamma_j$$

and thus for every  $m \geq k+1$

$$\max_{k+1 \leq n \leq m} 2^{-n} \gamma_n \leq 2^{-k} \gamma_k + \frac{1}{2} \max_{k+1 \leq j \leq m} 2^{-j} \gamma_j,$$

in particular for  $m \geq k+1$

$$2^{-m} \gamma_m \leq 2^{1-k} \gamma_k.$$

From this inequality and Lemma 4.2.2 we obtain by standard calculations

that for every  $\varepsilon > 0$

$$\frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2\pi} \int_{-1}^1 v(t) \log |\alpha'(t) + \varepsilon| dt \right\} \leq 2^{1-k} \gamma_k.$$

Letting  $\varepsilon \rightarrow 0$  we see that

$$\int_{-1}^1 v(t) \log \alpha'(t) dt > -\infty.$$

Hence  $\alpha \in S$ .

Lemma 2. Let  $w$  be of the form  $w(x) = \varphi(x)|x|^\varepsilon$  ( $\varepsilon > -1$ ,  $-1 \leq x \leq 1$ ) where  $\varphi$  is an even function of bounded variation with  $\varphi(1) > 0$ . Then for  $n = 1, 2, \dots$

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \frac{n + \frac{\varepsilon}{2} [1 + (-1)^{n+1}]}{[n + \frac{1}{2}(a_n + \varepsilon + 1)]^2 [n + \frac{1}{2}(a_{n-1} + \varepsilon - 1)]^2 - \frac{1}{4} b_n^2}$$

where

$$a_n = \int_{-1}^1 p_n^2(w, x) |x|^\varepsilon d\varphi(x)$$

and

$$b_n = \int_{-1}^1 p_{n-1}(w, x) p_n(w, x) |x|^\varepsilon d\varphi(x).$$

Proof. Since  $\gamma_{n-1}(w)/\gamma_n(w) = \gamma_{n-1}(cw)/\gamma_n(cw)$  for every  $c > 0$  we can suppose that  $\varphi(\pm 1) = 1$ .

We have

$$\int_{-1}^1 x [p_n^2(w, x)]' w(x) dx = 2 \int_{-1}^1 p_n(w, x) [n \gamma_n(w) x^n + \dots] w(x) dx = 2n.$$

On the other hand since  $w$  is even

$$\begin{aligned} \int_{-1}^1 x [p_n^2(w, x)]' w(x) dx &= 2 p_n^2(w, 1) - \int_{-1}^1 p_n^2(w, x) dx dw(x) = \\ &= 2 p_n^2(w, 1) - \int_{-1}^1 p_n^2(w, x) x dw(x). \end{aligned}$$

Thus

$$2n+1 = 2 p_n^2(w, 1) - \int_{-1}^1 p_n^2(w, x) x dw(x).$$

Because  $w(x) = \varphi(x)|x|^\varepsilon$  we obtain

$$\begin{aligned} \int_{-1}^1 p_n^2(w, x) x dw(x) &= \int_{-1}^1 p_n^2(w, x) |x|^\varepsilon d\varphi(x) + \\ &+ \varepsilon \int_{-1}^1 p_n^2(w, x) x \varphi(x) |x|^\varepsilon \text{sign } x dx = \int_{-1}^1 p_n^2(w, x) |x|^\varepsilon d\varphi(x) + \varepsilon. \end{aligned}$$

Putting (1) into (2) we finish the proof.

Theorem 3. Let  $\text{supp}(w) \subseteq [-1, 1]$ ,  $w$  be even and of bounded variation

with  $w(1) > 0$ . Let

$$(3) \quad \sup_{n \geq 1} \int_{-1}^1 p_n^2(w, x) |dw(x)| < \infty.$$

If  $w^{-\varepsilon} \in L(\tau)$  ( $\varepsilon > 0$ ,  $\tau \subset (-1, 1)$ ) then the sequence  $\{|p_n(w, t)|\}$  is bounded for almost every  $t \in \tau$ .

Proof. We apply Lemma 2 with  $\varepsilon = 0$ . Both  $\{a_n\}$  and  $\{b_n\}$  are bounded by (3). Thus

$$\frac{Y_{n-1}(w)}{Y_n(w)} = \frac{1}{2} + O\left(\frac{1}{n}\right).$$

Since  $w$  is even we obtain  $c_j^{0,1}(w) = O\left(\frac{1}{n}\right)$ . Now we use Theorem 6.2.2b.

Theorem 4. Let  $w(x) = \varphi(x)|x|^\varepsilon$  ( $\varepsilon > -1$ ,  $-1 \leq x \leq 1$ ). Let  $\varphi$  be even, continuous and positive and let  $\varphi'$  be also continuous. Then

$$\frac{Y_{n-1}(w)}{Y_n(w)} = \frac{1}{2} + (-1)^{n+1} \frac{\varepsilon}{4n} + o\left(\frac{1}{n}\right)$$

for  $n = 1, 2, \dots$ . If  $\varphi$  is constant then  $o\left(\frac{1}{n}\right)$  can be replaced by  $O\left(\frac{1}{n^2}\right)$ .

Proof. Let us use Lemma 2. By the conditions  $a_n = O(1)$  and  $b_n = O(1)$ .

Thus  $w \in M(0, 1)$ . Further

$$\begin{aligned} \left\{n + \frac{\varepsilon}{2} [1 + (-1)^{n+1}]\right\} \frac{Y_n}{Y_{n-1}} &= \\ &= 2n \left[1 + \frac{1}{2n} (a_n + \varepsilon + 1)\right]^2 \left[1 + \frac{1}{2n} (a_{n-1} + \varepsilon - 1)\right]^2 - b_n = \\ &= 2n \left[1 + \frac{1}{4n} (a_n + \varepsilon + 1) + O\left(\frac{1}{n^2}\right)\right] \cdot \left[1 + \frac{1}{4n} (a_{n-1} + \varepsilon - 1) + O\left(\frac{1}{n^2}\right)\right] - b_n = \\ &= 2n + \varepsilon + \frac{1}{2} (a_n + a_{n-1} - 2b_n) + O\left(\frac{1}{n}\right). \end{aligned}$$

Consequently

$$(1) \quad p_n^2(w, 1) = n + \frac{1+\varepsilon}{2} + \frac{1}{2} \int_{-1}^1 p_n^2(w, x) |x|^\varepsilon d\varphi(x)$$

for  $n = 0, 1, \dots$ . Now we shall consider another integral:

$$\begin{aligned} \int_{-1}^1 [p_n(w, x) p_{n-1}(w, x)]^\varepsilon w(x) dx &= \int_{-1}^1 p_n^\varepsilon(w, x) p_{n-1}^\varepsilon(w, x) w(x) dx = \\ &= \int_{-1}^1 p_{n-1}^\varepsilon(w, x) [n Y_n^\varepsilon(w) x^{n-1} + \dots] w(x) dx = n \frac{Y_n(w)}{Y_{n-1}(w)}. \end{aligned}$$

But

$$\int_{-1}^1 [p_n(w, x) p_{n-1}(w, x)]^\varepsilon w(x) dx = 2 p_n(w, 1) p_{n-1}^\varepsilon(w, 1) -$$

$$- \int_{-1}^1 p_n^\varepsilon(w, x) p_{n-1}^\varepsilon(w, x) dw(x)$$

and

$$\begin{aligned} \int_{-1}^1 p_n(w, x) p_{n-1}^\varepsilon(w, x) dw(x) &= \int_{-1}^1 p_n(w, x) p_{n-1}^\varepsilon(w, x) |x|^\varepsilon d\varphi(x) + \\ &+ \varepsilon \int_{-1}^1 p_n(w, x) p_{n-1}^\varepsilon(w, x) \frac{w(x)}{x} dx. \end{aligned}$$

If  $n$  is even then  $p_{n-1}^\varepsilon(w, x)x^{-1}$  is a polynomial of degree  $n-2$  and consequently the latter integral equals 0. If  $n$  is odd then  $p_{n-1}^\varepsilon(w, x)x^{-1}$  is a polynomial of degree  $n-1$ . Thus

$$\begin{aligned} \int_{-1}^1 p_{n-1}^\varepsilon(w, x) p_n(w, x) \frac{w(x)}{x} dx &= \int_{-1}^1 p_{n-1}^\varepsilon(w, x) [Y_n x^{n-1} + \dots] w(x) dx = \\ &= \frac{Y_n(w)}{Y_{n-1}(w)}. \end{aligned}$$

Hence for  $n = 1, 2, \dots$

$$\int_{-1}^1 p_n(w, x) p_{n-1}^\varepsilon(w, x) \frac{w(x)}{x} dx = \frac{1}{2} \frac{Y_n(w)}{Y_{n-1}(w)} [1 + (-1)^{n+1}].$$

Thus we obtain

$$\begin{aligned} (2) \quad \left\{n + \frac{\varepsilon}{2} [1 + (-1)^{n+1}]\right\} \frac{Y_n(w)}{Y_{n-1}(w)} &= \\ &= 2 p_n(w, 1) p_{n-1}^\varepsilon(w, 1) - \int_{-1}^1 p_n^\varepsilon(w, x) p_{n-1}^\varepsilon(w, x) |x|^\varepsilon d\varphi(x). \end{aligned}$$

Thus

$$\frac{y_n}{y_{n-1}} = 2 + \frac{\varepsilon(-1)^n}{n + \frac{\varepsilon}{2}[1+(-1)^{n+1}]} + O\left(\frac{|a_n + a_{n-1} - 2b_n|}{n}\right) + O\left(\frac{1}{n^2}\right).$$

Finally we obtain

$$\frac{y_{n-1}}{y_n} = \frac{1}{2} + (-1)^{n+1} \frac{\varepsilon}{4n} + O\left(\frac{|a_n + a_{n-1} - 2b_n|}{n}\right) + O\left(\frac{1}{n^2}\right).$$

If  $\varphi$  is constant then  $a_n = a_{n-1} = b_n = 0$ . If  $\varphi$  is not constant then we have to show that

$$(4) \quad \lim_{n \rightarrow \infty} (a_n + a_{n-1} - 2b_n) = 0.$$

By the recurrence formula

$$a_n = \frac{y_n}{y_{n+1}} b_{n+1} + \frac{y_{n-1}}{y_n} b_n.$$

Hence

$$a_n + a_{n-1} - 2b_n = \frac{y_n}{y_{n+1}} b_{n+1} + 2\left(\frac{y_{n-1}}{y_n} - 1\right)b_n + \frac{y_{n-2}}{y_{n-1}} b_{n-1}.$$

Since  $w \in M(0,1)$  if  $\lim_{n \rightarrow \infty} b_n$  exists and it is finite then (4) holds. But

$$b_n = \int_{-1}^1 p_n(w, x) p_{n-1}(w, x) \frac{\varphi'(x)}{\varphi(x)} w(x) dx.$$

By the conditions  $\varphi'/\varphi$  is continuous on  $[-1,1]$ . Using Theorem 4.2.13 we obtain

obtain

$$\lim_{n \rightarrow \infty} b_n = \frac{1}{\pi} \int_{-1}^1 \frac{t \varphi'(t)}{\varphi(t) \sqrt{1-t^2}} dt < \infty.$$

Consequently (4) is satisfied.

Theorem 5. Let  $\alpha$  be such that either  $\text{supp}(d\alpha) \subset [-1,1]$  or  $\alpha \in M(0,1)$ .

Let  $\tau \in [-1,1]$  and  $\varphi$  be defined by

$$\varphi(x) = \sup_{n \geq 0} p_n^2(d\alpha, x) \quad (x \in \tau).$$

Then for almost every  $x \in \tau$

$$\alpha'(x) \sqrt{1-x^2} \geq \frac{1}{\pi \varphi(x)},$$

In particular if  $\varphi(x)$  is finite for almost every  $x \in \tau$  then  $\alpha'(x) > 0$  for almost every  $x \in \tau$  and if  $\varphi(x) \leq K < \infty$  for almost every  $x \in \tau$  then

$$(5) \quad \alpha'(x) \sqrt{1-x^2} \geq \frac{1}{K\pi}$$

for almost every  $x \in \tau$ .

Proof. By the definition of  $\varphi$ ,  $n \setminus (d\alpha, x) \geq \varphi(x)^{-1}$  for  $x \in \tau$  and we apply Theorems 6.2.33 and 6.2.51.

Let us note that putting  $\alpha = \text{Tschebyshev weight}$  we see that the constant  $K$  in (5) is not exact.

Definition 6. Let  $\text{supp}(d\alpha) \subset [-1,1]$ . Let  $\mu = \mu_\alpha$  be the weight on the unit circumference associated with  $\alpha$  in the usual way:

$$\mu(\theta) = \begin{cases} \alpha(1) - \alpha(\cos \theta) & \text{for } 0 \leq \theta \leq \pi \\ \alpha(\cos \theta) - \alpha(-1) & \text{for } -\pi \leq \theta \leq 0. \end{cases}$$

Let  $\phi_n(d\mu, z) = z^n + \dots$  ( $n = 0, 1, \dots$ ) denote the corresponding system of orthogonal polynomials. It is known that the coefficients of  $\phi_n(d\mu, z)$  are real. (See e.g. Freud, §V.1) We put

$$a_n = -\phi_{n+1}(d\mu, 0).$$

Lemma 7.  $\alpha \in S$  iff  $\sum_{n=0}^{\infty} a_n^2 < \infty$ .

Proof. See Geronimus, §8.2.

Lemma 8. We have

$$\frac{y_n(d\alpha)}{y_{n+1}(d\alpha)} = \frac{1}{2} [(1-a_{2n-1})(1-a_{2n}^2)(1+a_{2n+1})]^{\frac{1}{2}}$$

and

$$\alpha_n(d\alpha) = \frac{1}{2} \left[ a_{2n-2}^{(1+\alpha)} a_{2n-1}^{(1-\alpha)} - a_{2n}^{(1-\alpha)} a_{2n-1}^{(1+\alpha)} \right]$$

for  $n = 1, 2, \dots$

Proof. Calculation. For the first relation see e. g. Geronimus, §9.1.

From Lemmas 7 and 8 we obtain

Theorem 9. Let  $\alpha \in S$ . Then

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha)^2 < \infty.$$

Remark 10. The converse of Theorem 9 is not true. Example: the Polleczeck weight.

Let us note that Geronimus has proved that if  $\sum_{n=0}^{\infty} |a_n| < \infty$  then  $\mu$  is absolutely continuous,  $\mu'$  is continuous and positive and  $|\varphi_n(d\mu, z)| \leq C < \infty$  for  $n = 1, 2, \dots$ ,  $-\pi \leq \theta \leq \pi$ ,  $z = e^{i\theta}$ . Here  $\varphi_n(d\mu, z)$  denotes the corresponding orthonormal polynomial. It is obvious that neither  $\alpha'(x) > 0$  for  $-1 \leq x \leq 1$  nor  $|p_n(d\alpha, x)| \leq C < \infty$  for  $n = 1, 2, \dots$ ;  $-1 \leq x \leq 1$  follows from  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ . Later we shall show that  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$  neither implies that  $\text{supp}(d\alpha) = [-1, 1]$  nor that  $\alpha$  is absolutely continuous but  $\text{supp}(\alpha') = [-1, 1]$  follows from  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ .

Theorem 11. Let  $\alpha \in S$ . Then the series

$$\sum_{k=1}^{\infty} (1-x^2)^k \lambda_{2k+1}^2(d\alpha, x) p_{2k}^2(d\alpha, x)$$

converges uniformly for  $x \in [-1, 1]$ .

Proof. By Theorem 3.1.8

$$(1-x^2)^k \lambda_{2k+1}^2(d\alpha, x) p_{2k}^2(d\alpha, x) \leq C [2^{-k} + \sum_{j=2^{-k}+1}^{2^k} c_j^{0,1}(d\alpha)^2]$$

and we apply Theorem 9.

Theorem 12. Let either  $\alpha \in M(0, 1)$  or  $\text{supp}(d\alpha) \subset [-1, 1]$ . Let  $n_1 < n_2 < \dots$  be such that  $\sum_{k=1}^{n-1} n_k < \infty$ . Then the series

$$\sum_{k=1}^{\infty} \lambda_{n_k}(d\alpha, x) p_{n_k}^2(d\alpha, x)$$

converges for almost every  $x \in \text{supp}(d\alpha)$ .

Proof. Use Theorems 3.3.7, 6.2.33, 6.2.51 and Beppo Levi's theorem.

Lemma 13. Let  $p_n(d\alpha, x) = \gamma_n(d\alpha)x^n + \mu_n(d\alpha)x^{n-1} + \dots$ . Then

$$\sum_{j=0}^{n-1} \alpha_j(d\alpha) = -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)}$$

for  $n = 1, 2, \dots$ , in particular,

$$\alpha_n(d\alpha) = \frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)} - \frac{\mu_{n+1}(d\alpha)}{\gamma_{n+1}(d\alpha)}.$$

Proof. We have

$$\begin{aligned} \sum_{k=0}^{n-1} \alpha_k(d\alpha) &= \int_{-\infty}^{\infty} x \lambda_n^{-1}(d\alpha, x) d\alpha(x) = \\ &= \sum_{k=1}^n \lambda_{kn}^{-1}(d\alpha) \int_{-\infty}^{\infty} x^2 \lambda_{kn}^{-1}(d\alpha, x) d\alpha(x) = \sum_{k=1}^n x_{kn}(d\alpha) = \\ &= -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)}. \end{aligned}$$

Definition 14. Let  $t \in \mathbb{R}$ . Then  $\delta_t$  denotes the unit mass concentrated at  $t$ , that is

$$\delta_t(x) = \begin{cases} 0 & \text{for } x < t \\ 1 & \text{for } x \geq t. \end{cases}$$

Lemma 15. Let  $\varepsilon > 0$ ,  $t \in \mathbb{R}$ ,  $\beta = \alpha + \varepsilon \delta_t$ . Then

$$(b) \quad p_n(d\beta, x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} \left[ p_n(d\alpha, x) - \frac{\varepsilon p_n(d\alpha, t) K_{n+1}^{(d\alpha, t, x)}}{1 + \varepsilon K_{n+1}^{(d\alpha, t, t)}} \right]$$

where

$$(7) \quad \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)} = 1 - \frac{\varepsilon p_n^2(d\alpha, t)}{1 + \varepsilon K_{n+1}^{(d\alpha, t, t)}},$$



further

$$(8) \quad \alpha_n(d\beta) = \alpha_n(d\alpha) + \epsilon \frac{Y_n(d\alpha)}{Y_{n+1}(d\alpha)} \frac{P_n(d\alpha, t) P_{n+1}(d\alpha, t)}{1 + \epsilon K_{n+1}(d\alpha, t, t)} - \epsilon \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \frac{P_{n-1}(d\alpha, t) P_n(d\alpha, t)}{1 + \epsilon K_n(d\alpha, t, t)}.$$

If  $\alpha(x) + \alpha(-x) = \text{const.}$   $t > 0$ ,  $\epsilon > 0$  and  $\beta = \alpha + \epsilon\delta_t + \epsilon\delta_{-t}$  then

$$P_n(d\beta, x) = \frac{Y_n(d\alpha)}{Y_n(d\beta)} [P_n(d\alpha, x) - \epsilon P_n(d\alpha, t) \cdot \frac{K_{n+1}(d\alpha, t, x) + (-1)^n K_{n+1}(d\alpha, -t, x)}{1 + \epsilon [K_{n+1}(d\alpha, t, t) + (-1)^n K_{n+1}(d\alpha, -t, t)]}]$$

and

$$\frac{Y_n^2(d\beta)}{Y_n(d\alpha)} = 1 - \frac{\epsilon P_n^2(d\alpha, t)}{\frac{1}{2} + \epsilon \sum_{k=0}^n P_k^2(d\alpha, t)} \quad k \equiv n \pmod{2}$$

Proof. We shall prove only the first part of the Lemma, the second one can be shown exactly in the same way. Let us note that both (7) and (8) follow from (6). If we multiply both sides of (6) by  $P_n(d\alpha, x)/d\alpha(x)$  and we integrate over  $\mathbb{R}$  then we get (7). Using Lemma 13 and comparing coefficients in (6) we obtain (8). Let us now prove (6). Develop  $P_n(d\beta, x)$  into a Fourier series in  $P_k(d\alpha, x)$ . Then

$$P_n(d\beta, x) = \int_{-\infty}^{\infty} P_n(d\beta, u) K_{n+1}(d\alpha, x, u) d\alpha(u) = \frac{Y_n(d\alpha)}{Y_n(d\beta)} P_n(d\alpha, x) - \epsilon P_n(d\beta, t) K_{n+1}(d\alpha, x, t).$$

Putting here  $x = t$  we obtain

$$(9) \quad P_n(d\beta, t) = \frac{Y_n(d\alpha)}{Y_n(d\beta)} P_n(d\alpha, t) [1 + \epsilon K_{n+1}(d\alpha, t, t)]^{-1}.$$

(6) follows from the above two formulas.

Lemma 16. Let  $\alpha \in M(0, 1)$ ,  $\epsilon > 0$ ,  $t \in \mathbb{R}$ ,  $\beta_t = \alpha + \epsilon\delta_t$ . Then

$$\lim_{n \rightarrow \infty} \frac{Y_n(d\beta_t)}{Y_n(d\alpha)} = \begin{cases} 1 & \text{for } t \in \text{supp}(d\alpha) \\ |\mu(t)|^{-1} & \text{for } t \notin \text{supp}(d\alpha), \end{cases}$$

the convergence is uniform for  $t \in \Delta \subset (-1, 1)$ , further

$$\lim_{n \rightarrow \infty} \alpha_n(d\beta_t) = 0$$

for every  $t \in \mathbb{R}$  and the convergence is uniform for  $t \in \Delta \subset (-1, 1)$ . If

$z \in \mathbb{C} \setminus \text{supp}(d\alpha) \setminus \{t\}$  then for  $t \notin \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \frac{P_n(d\beta_t, z)}{P_n(d\alpha, z)} = |\mu(t)| \left[ 1 - \frac{\sqrt{t^2 - 1}}{\mu(t)} \frac{\mu(z) - \mu(t)}{z - t} \right]$$

and for  $t \in \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \frac{P_n(d\beta_t, z)}{P_n(d\alpha, z)} = 1$$

uniformly for  $t \in \Delta \subset (-1, 1)$ . Furthermore

$$\lim_{n \rightarrow \infty} P_n(d\beta_t, t) P_n(d\alpha, t) = \begin{cases} 0 & \text{for } t \in \text{supp}(d\alpha) \\ \frac{2}{\epsilon} \sqrt{|t|^2 - 1} & \text{for } t \notin \text{supp}(d\alpha). \end{cases}$$

If  $x, t \in \text{supp}(d\alpha)$ ,  $x \neq t$  and the sequence  $\{P_k(d\alpha, x)\}$  is bounded then

$$(10) \quad \lim_{n \rightarrow \infty} [P_n(d\beta_t, x) - P_n(d\alpha, x)] = 0.$$

(10) holds uniformly for  $x \in \mathbb{R} \subset \text{supp}(d\alpha)$  if  $t \in \text{supp}(d\alpha) \setminus \mathbb{R}$  and

$\{P_k(d\alpha, x)\}$  is uniformly bounded for  $x \in \mathbb{R}$ . Finally,  $\beta_t \in M(0, 1)$  for each  $t \in \mathbb{R}$ .

Proof. The Lemma follows immediately from Lemma 15, Theorems 4.1.11,

4.1.13, 4.1.14 and (9).

Let us note that all the limits in Lemma 16 - except for one - are independent of  $\epsilon$ .

Lemma 17. Let  $\alpha \in M(0,1)$  and

$$\sum_{j=0}^{\infty} c_j^{0,1}(da) < \infty.$$

Let for  $0 \leq k \leq n$   $R_{n,k}(da, x)$  be as in 3.1 (2)-(3). Then

$$\lim_{n \rightarrow \infty} R_{n,k}(da, x) = 0$$

uniformly for  $x \in \Delta \subset (-1,1)$ .

Proof. By Theorem 3.1.12 the sequence  $\{p_k(da, x)\}$  is uniformly bounded

for  $x \in \Delta \subset (-1,1)$ . Now we apply Corollary 3.1.5.

Lemma 18. Let  $B$  be a function on  $[0, \pi]$ . Let

$$\varphi_n(x) = \cos[n\theta + B(\theta)] \quad (x = \cos \theta)$$

for  $n = 0, 1, \dots$ . Then for  $1 \leq k \leq n$

$$\varphi_n(x) = \varphi_k(x) U_{n-k}(\varphi_k(x)) U_{n-k-1}(x)$$

$(-1 \leq x \leq 1)$ .

Proof. Apply Theorem 3.1.1.

Lemma 19. Let  $\alpha \in M(0,1)$  and

$$\sum_{j=0}^{\infty} c_j^{0,1}(da) < \infty.$$

Suppose that there exist three functions  $A, B$  and  $C$  on  $[0, \pi]$  and a se-

quence  $n_1 < n_2 < \dots < n_k \rightarrow \infty$  such that

$$(11) \quad \lim_{k \rightarrow \infty} [C(\theta) p_{n_k}(da, x) - \varphi_{n_k}(x)] = 0$$

and

$$(12) \quad \lim_{k \rightarrow \infty} [C(\theta) p_{n_k-1}(da, x) - \varphi_{n_k-1}(x)] = 0$$

where  $x = \cos \theta$ ,  $\varphi_n(x) = A(\theta) \cos[n\theta + B(\theta)]$ ,  $-1 < x < 1$ ,  $C(\theta) < \infty$ . Then

$$(13) \quad \lim_{n \rightarrow \infty} [C(\theta) p_n(da, x) - \varphi_n(x)] = 0.$$

If the convergence in (11) and (12) is uniform for  $x \in \Delta \subset (-1,1)$  and  $C(\theta)$

is uniformly bounded for  $x \in \Delta$  then (13) holds uniformly for  $x \in \Delta$ .

Proof. By Theorem 3.1.1 and Lemma 18

$$|C p_n - \varphi_n| \leq v |C p_k - \varphi_k| + |C p_{k-1} - \varphi_{k-1}| + C R_{n,k}$$

for  $1 \leq k \leq n$  where  $v$  is the Tschelyshev weight. For a fixed  $n$  let

$k = k(n)$  be defined by  $k = \max\{n_j : n_j \leq n\}$ . Then  $\lim_{n \rightarrow \infty} k = \infty$ . Now we use

Lemma 17.

Recall that the function  $\Gamma$  has been defined in Definition 4.2.4.

Theorem 20. Let  $\text{supp}(da) \subset [-1,1]$  and

$$\sum_{j=0}^{\infty} c_j^{0,1}(da) < \infty.$$

Then for almost every  $x \in [-1,1]$

$$(14) \quad \lim_{n \rightarrow \infty} [\sqrt{\alpha'(x)} \sqrt{1-x^2} p_n(da, x) - \sqrt{\frac{2}{\pi}} \cos(n\theta - \Gamma(\theta))] = 0$$

where  $x = \cos \theta$ .

Proof. By Theorem 1  $\alpha \in S$ . Thus the Theorem follows from Lemmas 19 and

4.2.5.

Lemma 21. We have

$$\sum_{k=0}^n p_k^2(v, x) = \frac{n+1}{2\pi} + \frac{1}{4\pi} \frac{U_{2n+1}(x)}{x}$$

$k \equiv n \pmod{2}$

and for  $t > 1$

$$\frac{p_n^2(v, t)}{1 + \sum_{k=0}^n p_k^2(v, t)} = \frac{4\sqrt{t^2-1}}{t^2} + O(1/n) t^{-2n}$$

where  $|O(1)| \leq C$  uniformly for  $1 + \epsilon \leq t < \infty$ .

Proof. Calculation.

Theorem 22. Let  $\alpha$  be the Tschebyshev weight,  $t > 1$ ,  $\beta = \alpha + \delta_t + \delta_{-t}$ .

Then

$$Y_n(d\beta) = \frac{2^{n-1}}{\sqrt{\pi}} [\rho(t)^{-2} + O(1) n \rho(t)^{-2n}]$$

for  $n = 1, 2, \dots$ .

Proof. Apply Lemmas 15 and 21.

Theorem 22 is interesting because of the following

Corollary 23. Let  $C > 0$  and  $\varepsilon > 0$ . Then there exists a weight

$\alpha \in M(0, 1)$  such that  $[-C, C] \subset \Delta(d\alpha)$  and  $c_j^{0,1}(d\alpha) = O(\varepsilon^j)$ , in particular  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ .

Proof. For the weight  $\beta$  constructed in Theorem 22  $\alpha_j(d\alpha) = 0$  for

$j = 0, 1, \dots$ .

Lemma 24. Let  $\alpha \in M(0, 1)$ ,  $\varepsilon > 0$ ,  $-1 < t < 1$ ,  $\beta = \alpha + \varepsilon \delta_t$ . If

$$\limsup_{n \rightarrow \infty} \sum_{j=n}^{2n} j c_j^{0,1}(d\alpha)^2 < \infty$$

then

$$\alpha_n(d\beta) = \alpha_n(d\alpha) + O\left(\frac{1}{n}\right)$$

and

$$\frac{Y_{n-1}(d\beta)}{Y_n(d\beta)} = \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} + O\left(\frac{1}{n}\right)$$

for  $n = 1, 2, \dots$ .

Proof. The Lemma follows from Theorem 3.1.8 and Lemma 15. It might be interesting to remark that the estimates do not depend on  $\varepsilon$ .

By Lemma 16 from  $\alpha \in M(0, 1)$  follows that  $\alpha + \varepsilon \delta_t \in M(0, 1)$  for every  $\varepsilon > 0$ ,  $t \in \mathbb{R}$ . Hence  $\beta = \alpha + \sum_{k=1}^N \varepsilon_k \delta_{t_k}$  also belongs to  $M(0, 1)$  and by repeating application of the previous results we obtain asymptotics and estimates for  $p_n(d\beta, z)$ ,  $c_j^{0,1}(d\beta)$ ,  $\lambda_n(d\beta, z)$  etc. Let us mention two results.

Theorem 25. Let  $\alpha \in M(0, 1)$ ,  $t_k \in \mathbb{R}$ ,  $\varepsilon_k > 0$  for  $k = 1, 2, \dots, N$ . Let  $\beta = \alpha + \sum_{k=1}^N \varepsilon_k \delta_{t_k}$ . Then  $\beta \in M(0, 1)$ .

$$\lim_{n \rightarrow \infty} \frac{Y_n(d\beta)}{Y_n(d\alpha)} = \prod_{k=1}^N \frac{|\rho(t_k)|^{-1}}{t_k \supp(d\alpha)}$$

and for every  $z \in \mathbb{C} \setminus \supp(d\beta)$

$$\lim_{n \rightarrow \infty} \frac{p_n(d\beta, z)}{p_n(d\alpha, z)} = \prod_{k=1}^N \frac{(|\rho(t_k)| |1 - \frac{\sqrt{t_k^2 - 1}}{t_k} \frac{\rho(z) - \rho(t_k)}{z - t_k}|)}{t_k \supp(d\alpha)}$$

Theorem 26. Let  $\supp(d\alpha) \subset [-1, 1]$ ,  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ ,  $\beta$  be defined as in Theorem 25 with  $t_k \in (-1, 1)$  for  $k = 1, 2, \dots, N$ . Then for almost every

$x \in [-1, 1]$  the asymptotic formula (14) holds if we replace there  $p_n(d\alpha, x)$  by  $p_n(d\beta, x)$ .

We suggest the reader combine these results with those of Sections

4.1, 4.2 and 6.1.

Theorem 27. Let

$$\sum_{j=0}^{\infty} j c_j^{(0,1)}(d\alpha) < \infty.$$

Then there exists a positive number  $K = K(d\alpha)$  such that

$$\sqrt{1-x} |p_n(d\alpha, x)| \leq K$$

for  $-1 \leq x \leq 1$  and  $n = 0, 1, \dots$ . Further

$$\alpha'(x) \geq \frac{1}{K^2} \sqrt{1-x^2}$$

for almost every  $x \in [-1, 1]$ , in particular,  $\alpha' \in S$ .

Proof. By an inequality of S. Bernstein

$$\max_{|x| \leq 1} |\pi_m(x)| \leq m \max_{|x| \leq 1} |\sqrt{1-x^2} \pi_m(x)|$$

for every  $\pi_m$  if  $m \geq 1$ . Let  $2 \leq k \leq n$ . Then we have by Corollary 3.5.1



$$\max_{|x| \leq 1} |\sqrt{1-x^2} p_n(da, x)| \leq \max_{|x| \leq 1} [|p_k(da, x)| + |p_{k-1}(da, x)|] + 2 \sum_{j=k-1}^n |c_j^{(0,1)}(da)| \max_{|x| \leq 1} |\sqrt{1-x^2} p_j(da, x)|,$$

or in short

$$A_n \leq C(k) + 2 \sum_{j=k+1}^n |c_j| A_j$$

where  $c_j = c_j^{(0,1)}(da)$ ,  $A_j = \max_{|x| \leq 1} |\sqrt{1-x^2} p_j(da, x)|$  and  $C(k)$  is a constant depending on  $k$  and  $da$ . Let now  $k$  be so large that  $\sum_{j=k+1}^{\infty} |c_j| < \frac{1}{4}$ .

Then

$$A_{k+1} \leq 2 C(k).$$

Suppose that  $A_m \leq 2 C(k)$  for  $k < m < n$ . If  $A_n > 2 C(k)$  then  $A_m \leq 2 C(k)$

$< A_n$ ,  $m = k+1, \dots, n-1$ , so

$$A_n \leq C(k) + 2 \sum_{j=k+1}^n |c_j| A_j < C(k) + A_n 2 \sum_{j=k+1}^n |c_j| < C(k) + \frac{1}{4} A_n,$$

that is  $A_n < 2 C(k)$ . This contradiction shows that  $A_n \leq 2 C(k)$  for  $n > k$ .

To finish the proof we apply Theorem 5.

Remark 28. If we put  $da(x) = \sqrt{1-x^2} dx$ ,  $\text{supp}(da) = [-1, 1]$  then  $c_j^{(0,1)}(da) = 0$

for every  $j = 0, 1, \dots$ ,  $\sigma'(x) = \sqrt{1-x^2}$  and for each  $\tau \in [-1, 1]$

$$\max_{x \in \tau} |\sqrt{1-x^2} p_n(da, x)| = \sqrt{\frac{2}{\pi}}$$

if  $n \geq n_1(\tau)$  ( $n_1(-1, 1) = 0$ ). Hence Theorem 27 - except for the constant - cannot be improved.

From Theorems 5 and 3.1.12 we obtain

Theorem 29. Let  $\sum_{j=0}^{\infty} c_j^{(0,1)}(da) < \infty$ . Then for each  $\tau \in (-1, 1)$  there exists a number  $K = K(\tau, da) > 0$  such that  $\sigma'(x) \geq K$  for almost every  $x \in \tau$ . In

particular if  $\sigma'$  is continuous at  $\tau \in (-1, 1)$  and  $\sigma'(\tau) = 0$  then

$$\sum_{j=0}^{\infty} c_j^{(0,1)}(da) = \infty.$$

Corollary 30. If  $\sum_{j=0}^{\infty} c_j^{(0,1)}(da) < \infty$  then  $\text{supp}(\sigma') = [-1, 1]$ .

Proof. Theorems 3, 3.7 and 29.

Theorem 31. Let  $\sigma$  be such that either  $\text{supp}(da) \subset [-1, 1]$  or  $\sigma \in M(0, 1)$ .

Let  $p \geq 2$  and  $w(\geq 0) \in L^1(-1, 1)$ . Then from

$$\sup_{n \geq 0} \int_{-1}^1 |p_n(da, x)|^p w(x) dx < \infty$$

follows

$$(15) \quad \int_{-1}^1 |\sigma'(x) \sqrt{1-x^2}|^{\frac{p}{2}} w(x) dx < \infty.$$

Proof. Since  $p/2 \geq 1$  we have

$$\|n^{-\frac{1}{p-1}} \lambda_n^{-1}(da)\|_{w, \frac{p}{2}} \leq \frac{1}{n} \sum_{k=0}^{n-1} \|p_k^2(da)\|_{w, \frac{p}{2}} \leq \sup_{n \geq 0} \|p_n^2(da)\|_{w, \frac{p}{2}}.$$

Because  $\int \liminf_n |f| \leq \limsup_n \int |f|$  the theorem follows from Theorems 6, 2.33 and 6.2.51.

Theorem 32. Let  $\sigma \in S$ ,  $0 < p < \infty$ ,  $w(\geq 0) \in L^1(-1, 1)$ . Then

$$\liminf_n \int_{-1}^1 |p_n(da, x)|^p w(x) dx < \infty$$

implies (15).

Proof. First let  $0 < p \leq 2$ . Then we shall use Lemma 4.2.5. Let  $N > 0$  and  $\varphi_N$  be defined by

$$\varphi_N(t) = \min \{N, w(\cos t) \sin t |\sin t \sigma'(\cos t)|^{\frac{p}{2}}\}$$

for  $0 \leq t \leq \pi$ . Then

$$\begin{aligned} & \int_0^\pi \left| \int_0^\pi \sqrt{\frac{2}{\pi}} \cos[nt - \Gamma(t)] \varphi_N(t) dt \right|^p \leq \left| \int_0^\pi \sqrt{\frac{2}{\pi}} \cos[nt - \Gamma(t)] \varphi_N(t) dt \right|^p \leq \\ & \leq 2^p \int_0^\pi |p_n(da, \cos t) \sqrt{\sigma'(\cos t) \sin t} - \sqrt{\frac{2}{\pi}} \cos[nt - \Gamma(t)]|^p \varphi_N(t) dt + \\ & + 2^p \left| \int_0^\pi |p_n(da, \cos t) \sqrt{\sigma'(\cos t) \sin t}|^p \varphi_N(t) dt \right|^p \leq \\ & \leq 2^p N^{\frac{p}{2}} \int_0^\pi |p_n(da, \cos t) \sqrt{\sigma'(\cos t) \sin t}|^p \varphi_N(t) dt + \\ & + 2^p \int_0^\pi \cos[nt - \Gamma(t)]^2 dt + 2^p \int_0^\pi |p_n(da, x)|^p w(x) dx \end{aligned}$$



By Lemma 4.2.5 the first term in the right side converges to 0 when  $n \rightarrow \infty$ .

The limit inferior of the second term is finite. By the Riemann-Lebesgue

lemma the left side converges to

$$\left( \frac{1}{\pi} \int_0^\pi \varphi_N(t) dt \right)^{\frac{1}{p}}$$

when  $n \rightarrow \infty$ . Thus by the above inequalities

$$\frac{1}{\pi} \int_0^\pi \varphi_N(t) dt \leq 2 \liminf_{N \rightarrow \infty} \int_{-1}^1 |p_n(d\alpha, x)|^p w(x) dx.$$

By Beppo Levi's theorem  $\lim_{N \rightarrow \infty} \varphi_N \in L^1(0, \pi)$  that is (15) holds. If  $p > 2$

then for  $N > 0$

$$\begin{aligned} & \int_{-1}^1 p_n^2(d\alpha, x) |\alpha'(x) \sqrt{1-x^2} + N^{-1}|^{\frac{p}{2}} w_N(x) \alpha'(x) \sqrt{1-x^2} dx \\ & \cdot \left| \int_{-1}^1 [\alpha'(x) \sqrt{1-x^2} + N^{-1}]^{\frac{p}{2}} w_N(x) dx \right|^{\frac{2-p}{2}} \leq \\ & \leq \left\{ \int_{-1}^1 |p_n(d\alpha, x)|^p w(x) dx \right\}^{\frac{1}{p}} \end{aligned}$$

where  $w_N(x) = \min\{N, w(x)\}$ . Letting  $n \rightarrow \infty$  we obtain from Lemma 4.2.5 and from Riemann-Lebesgue's lemma that

$$\begin{aligned} & \pi^{-p} \left\{ \int_{-1}^1 [\alpha'(x) \sqrt{1-x^2} + N^{-1}]^{\frac{p}{2}} w_N(x) dx \right\}^2 \leq \\ & \leq \liminf_{n \rightarrow \infty} \int_{-1}^1 |p_n(d\alpha, x)|^p w(x) dx. \end{aligned}$$

Hence again by Beppo Levi's theorem (15) holds.

**Theorem 33.** Let  $w \in M(0, 1)$ ,  $\text{supp}(w) = [-1, 1]$ ,  $w$  be Riemann integrable on  $[-1, 1]$ . Let  $g(\geq 0)$  be almost everywhere continuous on  $[-1, 1]$  and  $p \geq 2$ . Then

$$\liminf_{n \rightarrow \infty} \int_{-1}^1 |p_n(w, x)|^p g(x) dx < \infty$$

implies

$$\int_{-1}^1 [w(x) \sqrt{1-x^2}]^{\frac{p}{2}} g(x) dx < \infty$$

Proof. In the conditions the function  $\varphi_N$  defined by

$$\varphi_N(x) = [w(x) \sqrt{1-x^2} + N^{-1}]^{\frac{p}{2}} \min\{N, g(x)\} \sqrt{1-x^2}$$

( $N > 0$ ) is Riemann integrable for each  $N > 0$ . Now we can repeat the

second part of the proof of Theorem 32. Applying Theorem 4.2.14 we obtain

the theorem.

**Theorem 34.** Let  $\alpha \in M(0, 1)$  and

$$\sum_{j=0}^{\infty} c_{0,1}^j(d\alpha) < \infty.$$

Then

$$\lim_{k \rightarrow \infty} [p_k^2(d\alpha, x) - p_{k-1}^2(d\alpha, x) p_{k+1}(d\alpha, x)] = \frac{2\sqrt{1-x^2}}{\pi \alpha'(x)}$$

for almost every  $x \in \text{supp}(d\alpha)$ .

Proof. Let  $0 \leq k \leq n$  and  $\Delta \subset (-1, 1)$ . Then by Lemma 17

$$p_n(d\alpha, x) = U_{n-k}(x) p_k(d\alpha, x) - U_{n-k-1}(x) p_{k-1}(d\alpha, x) + \sigma(1)$$

where  $\lim_{n \geq k \rightarrow \infty} \sigma(1) = 0$  uniformly for  $x \in \Delta$ . By Theorem 3.1.12 the sequence

$\{|p_k(d\alpha, x)|\}$  is uniformly bounded for  $x \in \Delta$ . Hence

$$\begin{aligned} p_n^2(d\alpha, x) &= U_{n-k}^2(x) p_k^2(d\alpha, x) + U_{n-k-1}^2(x) p_{k-1}^2(d\alpha, x) - \\ &- 2p_{k-1}(d\alpha, x) p_k(d\alpha, x) U_{n-k-1}(x) U_{n-k}(x) + \sigma(1). \end{aligned}$$

Thus for  $m > k$

$$\begin{aligned} \lambda_{m+1}^{-1}(d\alpha, x) &= \sum_{j=0}^k p_k^2(d\alpha, x) + \sum_{j=1}^{m-k} U_j^2(x) p_k^2(d\alpha, x) + \\ &+ \sum_{j=0}^{m-k-1} U_j^2(x) p_{k-1}^2(d\alpha, x) - \\ &- 2 \sum_{j=0}^{m-k-1} U_j(x) U_{j+1}(x) p_{k-1}(d\alpha, x) p_k(d\alpha, x) + m \sigma(1). \end{aligned}$$

Let us divide this formula by  $m$  and let  $m \rightarrow \infty$ . We obtain from Theorems 29, 6, 2.52 and Corollary 6.2.53 that

$$\frac{1}{\pi \sigma'(x) \sqrt{1-x^2}} = \frac{1}{2(1-x^2)} [p_k^2(d\sigma, x) + p_{k-1}^2(d\sigma, x)] - \frac{T_1(x)}{1-x^2} p_{k-1}(d\sigma, x) p_k(d\sigma, x) + \sigma(1)$$

for almost every  $x \in \Delta$ , that is

$$(16) \quad \frac{2\sqrt{1-x^2}}{\pi \sigma'(x)} = p_k^2(d\sigma, x) + p_{k-1}^2(d\sigma, x) - 2xp_{k-1}(d\sigma, x)p_k(d\sigma, x) + \sigma(1)$$

for almost every  $x \in \Delta$ . By Theorem 3.1.12 and by the recurrence formula

$$\lim_{k \rightarrow \infty} |2xp_{k-1}(d\sigma, x)p_k(d\sigma, x) - p_k^2(d\sigma, x) - p_{k-2}(d\sigma, x)p_k(d\sigma, x)| = 0$$

uniformly for  $x \in \Delta$ . Hence

$$(17) \quad \frac{2\sqrt{1-x^2}}{\pi \sigma'(x)} = p_{k-1}^2(d\sigma, x) - p_{k-2}(d\sigma, x)p_k(d\sigma, x) + \sigma(1)$$

for almost every  $x \in \Delta$  where  $\lim_{k \rightarrow \infty} \sigma(1) = 0$  uniformly for  $x \in \Delta$ . Since  $\Delta \subset (-1, 1)$  is arbitrary the Theorem follows.

Let us note that the determinant

$$\begin{vmatrix} p_k(d\sigma, x) & p_{k-1}(d\sigma, x) \\ p_{k+1}(d\sigma, x) & p_k(d\sigma, x) \end{vmatrix} = D_k(d\sigma, x)$$

is a rather famous expression, its positivity has been investigated by several authors. (See Szegő, Problems and exercises.) So far  $D_k(d\sigma, x)$  has been considered for the classical weights. From Theorem 29 and (17) we obtain the following

Corollary 35. Let  $\sum_{j=0}^{\infty} c_j^{0,1}(d\sigma) < \infty$  and  $\Delta \subset (-1, 1)$ . Then there exists a number  $N = N(\sigma, \Delta) > 0$  such that for each  $k \geq N$   $D_k(d\sigma, x) > 0$  whenever  $x \in \Delta$ .

The example of the Tschubyshev polynomials shows that  $\Delta$  cannot be replaced by  $[-1, 1]$  in Corollary 35.

Corollary 36. If  $\sum_{j=0}^{\infty} c_j^{0,1}(d\sigma) < \infty$  then

$$\limsup_{n \rightarrow \infty} \sigma'(x) \sqrt{1-x^2} p_n^2(d\sigma, x) = \frac{2}{\pi}$$

for almost every  $x \in \text{supp}(d\sigma)$ .

Proof. By Theorems 29 and 6.2.51

$$\limsup_{n \rightarrow \infty} \sigma'(x) \sqrt{1-x^2} p_n^2(d\sigma, x) \geq \frac{2}{\pi}$$

for almost every  $x \in \text{supp}(d\sigma)$ . On the other hand by (16)

$$p_{k-1}(d\sigma, x) = x p_k(d\sigma, x) \pm [(x^2-1)p_k^2(d\sigma, x) + \frac{2\sqrt{1-x^2}}{\pi \sigma'(x)} + \sigma(1)]^{\frac{1}{2}}$$

for almost every  $x \in [-1, 1]$ , that is by Theorem 3.3.7 for almost every  $x \in \text{supp}(d\sigma)$ . Hence

$$(1-x^2)p_n^2(d\sigma, x) \leq \frac{2\sqrt{1-x^2}}{\pi \sigma'(x)} + \sigma(1).$$

Letting  $n \rightarrow \infty$  and using Theorem 29 we obtain the Corollary.

Although the proof of the following Theorem is very simple it is one of our strongest results.

Theorem 37. Let  $\sigma \in S$ . Then

$$\lim_{n \rightarrow \infty} \frac{p_n^2(d\sigma, x)}{n} = 0$$

for almost every  $x \in [-1, 1]$ .

Proof. By Corollary 3.1.5,

$$(18) \quad (1-x^2)p_n^2(d\sigma, x) \leq 2(|p_k(d\sigma, x)| + |p_{k-1}(d\sigma, x)|)^2 + 8n \sum_{j=k-1}^{\infty} c_j^{0,1}(d\sigma) p_j^2(d\sigma, x).$$

By Theorem 9 and Beppo Levi's theorem

$$\sum_{j=0}^{\infty} c_j^{0,1} (d\alpha)^2 p_j^2(d\alpha, x) < \infty$$

for almost every  $x \in [-1, 1]$ . Dividing both sides of (18) by  $n$  and first letting  $n \rightarrow \infty$  and then  $k \rightarrow \infty$  we finish the proof.

Lemma 38. Let  $\phi_{2n}(dw, z)$  be defined as in Theorem 3.1.15. If

$$\sum_{j=0}^{\infty} c_j^{0,1} (d\alpha) < \infty \text{ then}$$

$$(19) \quad \phi(d\alpha, z) = \lim_{n \rightarrow \infty} \phi_{2n}(d\alpha, z)$$

$(z=e^{i\theta})$  exists for each  $\theta \in (0, 2\pi) \setminus \{\pi\}$  and the convergence is uniform for

$\theta \in \tau \subset (0, 2\pi) \setminus \{\pi\}$ , in particular,  $\phi(d\alpha, e^{i\theta})$  is continuous on  $(0, 2\pi) \setminus \{\pi\}$ .

If  $\sum_{j=0}^{\infty} j c_j^{0,1} (d\alpha) < \infty$  then (19) exists for each  $|z| \leq 1$ , the convergence is uniform in the unit disc,  $\phi(d\alpha, z)$  is analytic for  $|z| < 1$  and continuous for  $|z| \leq 1$ .

Proof. Apply Theorems 3.1.12, 27 and Bernstein's inequality

$$\max_{|x| \leq 1} |\pi'(x)| \leq m \max_{|x| \leq 1} \sqrt{1-x^2} \pi(x) \quad (m \geq 1).$$

Lemma 39. Let  $\phi_{2n}(d\alpha, z)$  be as in Theorem 3.1.15. Let  $\sum_{j=0}^{\infty} j c_j^{0,1} (d\alpha) < \infty$ .

Then

$$\lim_{n \rightarrow \infty} z^{2n} \phi_{2n}(d\alpha, z^{-1}) = 0$$

uniformly for  $|z| \leq 1 - \varepsilon$  ( $\varepsilon > 0$ ).

Proof. We have

$$z^{2n} \phi_{2n}(d\alpha, z^{-1}) = \sum_{j=0}^n a_j(d\alpha, \frac{z+z^{-1}}{2}) z^{2n-j}.$$

By Theorem 27

$$\sqrt{1-x^2} |p_j(d\alpha, x)| \leq K$$

for  $-1 \leq x \leq 1$ ,  $j = 0, 1, \dots$ . Hence

$$|(1-z^2)z^j p_j(d\alpha, \frac{z+z^{-1}}{2})| \leq C$$

for  $|z| \leq 1$ ,  $j = 0, 1, \dots$ . Consequently

$$|z^{2n} \phi_{2n}(d\alpha, z^{-1})| \leq C(1-|z|)^{2n-1}.$$

$$\begin{aligned} & \sum_{j=0}^n \left\{ (1-2 \frac{y_j-1}{y_j}) |z|^{2n-2j} + 2|\alpha_{j-1}| |z|^{2n-2j+1} + \right. \\ & \quad \left. + |1-2 \frac{y_j-2}{y_j}| |z|^{2n-2j+2} \right\} \end{aligned}$$

which converges uniformly to 0 if  $|z| \leq 1 - \varepsilon$ .

Theorem 40. Let  $\alpha \in M(0, 1)$  and  $\sum_{j=0}^{\infty} c_j^{0,1} (d\alpha) < \infty$ . Then  $d\alpha$  can be written in the form

$$d\alpha(t) = \alpha'(t) dt + d\alpha_j(t)$$

where  $\alpha'$  is continuous and positive in  $(-1, 1)$ ,  $\text{supp}(\alpha') = \{-1, 1\}$  and  $\alpha_j(t)$  is constant for  $-1 < t < 1$ . Further

$$(20) \quad \lim_{n \rightarrow \infty} \left[ \sin \theta p_n(d\alpha, \cos \theta) - \left[ \frac{2}{\pi} \frac{\sin \theta}{\alpha'(\cos \theta)} \right]^{\frac{1}{2}} \sin[(n+1)\theta - \varphi(\theta)] \right] = 0$$

uniformly for  $\theta \in \tau \subset (0, \pi)$ , where  $\varphi(\theta) = \arg \alpha(d\alpha, e^{i\theta})$ . (See (19).) is continuous in  $(0, \pi)$ .  $\alpha'$  can be calculated by the formula

$$\frac{2}{\pi} \frac{\sin \theta}{\alpha'(\cos \theta)} = |\alpha(d\alpha, e^{i\theta})|^2 = \lim_{n \rightarrow \infty} p_n^2(d\alpha, x) - p_{n-1}^2(d\alpha, x) p_{n+1}^2(d\alpha, x)$$

( $x = \cos \theta$ ). If  $\sum_{j=0}^{\infty} j c_j^{0,1} (d\alpha) < \infty$  then

$$\lim_{n \rightarrow \infty} \{ \sin \theta p_n(d\alpha, \cos \theta) - \psi(\theta) \sin[(n+1)\theta - \varphi(\theta)] \} = 0$$

uniformly for  $\theta \in [0, \pi]$ . Here  $\psi(\theta) = |\alpha(d\alpha, e^{i\theta})|$  and  $\varphi(\theta) = \arg \alpha(d\alpha, e^{i\theta})$  are continuous functions on  $[0, \pi]$ .

AD-A038 959

WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER  
ORTHOGONAL POLYNOMIALS.(U)

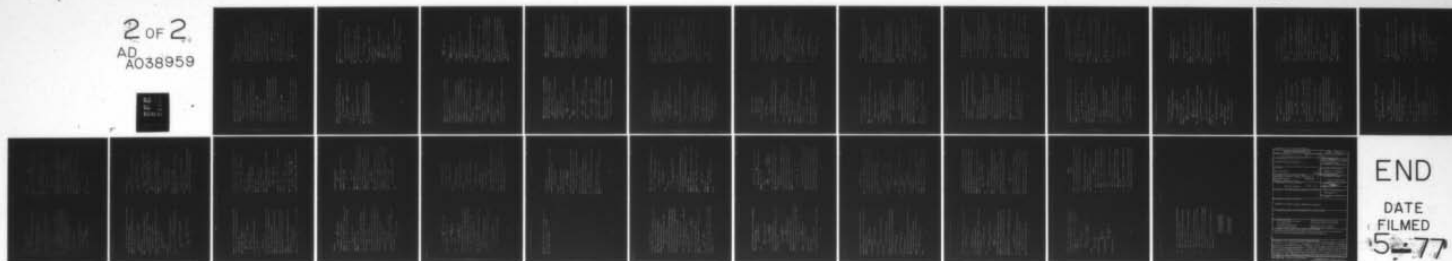
F/G 12/1

FEB 77 P 6 NEVAI  
MRC-TSR-1726

DAAG29-75-C-0024  
NL

UNCLASSIFIED

2 OF 2  
AD  
A038959



END

DATE  
FILMED

5-77



Proof. Let first  $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ . Then by Lemma 38 and Theorem 3.1.15

$$\lim_{n \rightarrow \infty} \{ \sin \theta_p(d\alpha, \cos \theta) - |\phi(d\alpha, e^{i\theta})| \sin[(n+1)\theta - \arg \phi(d\alpha, e^{i\theta})] \} = 0$$

uniformly for  $\theta \in (0, \pi)$ . Now let us calculate  $|\phi(d\alpha, e^{i\theta})|$ . We have

$$(1-x^2) \frac{1}{n \chi_n(d\alpha, x)} = |\phi(d\alpha, e^{i\theta})|^2.$$

$$- \frac{1}{2n} \sum_{k=0}^n \{1 - \cos[2(k+1)\theta - 2 \arg \phi(d\alpha, e^{i\theta})]\} + \sigma(1)$$

( $x = \cos \theta$ ). Here the right side converges to  $\frac{1}{2} |\phi(d\alpha, e^{i\theta})|^2$  when  $n \rightarrow \infty$

and the convergence is uniform for  $\theta \in (0, \pi)$ . By Theorem 6.2.54

$$\liminf_{n \rightarrow \infty} (1-x) \frac{1}{n \chi_n(d\alpha, x)} = \frac{\sqrt{1-x^2}}{\pi \alpha'(x)}$$

for almost every  $x \in \text{supp}(d\alpha)$ . Hence

$$(21) \quad |\phi(d\alpha, e^{i\theta})|^2 = \frac{2\sqrt{1-x^2}}{\pi \alpha'(x)}$$

for almost every  $\theta \in (0, \pi)$ . Consequently  $[\alpha']^{-1} \sqrt{1-x^2}$  is equivalent

to a continuous function. By Theorem 29  $\alpha'(x) \geq k > 0$  for almost every

$\theta \in (0, \pi)$ . Thus  $\alpha'(\cos \theta)$  and  $|\phi(d\alpha, e^{i\theta})|$  are continuous and positive

in  $(0, \pi)$  and (21) holds for each  $\theta \in (0, \pi)$ . Hence (20) holds uniformly for

$\theta \in \tau \subset (0, \pi)$ . From (20) we obtain

$$(22) \quad \lim_{n \rightarrow \infty} \frac{1}{n \chi_n(d\alpha, x)} = \frac{1}{\pi \alpha'(x) \sqrt{1-x^2}}$$

uniformly for  $x \in \Delta \subset (-1, 1)$ . Thus

$$\limsup_{n \rightarrow \infty} p_n^2(d\alpha, x) > 0$$

for every  $x \in (-1, 1)$ , that is  $\alpha_j$  must be constant in  $(-1, 1)$ . Now we shall

show that  $\alpha$  has no singular component. Because  $\alpha_j$  is constant in  $(-1, 1)$

for every  $\Delta \subset (-1, 1)$   $1_{\Delta} \alpha' v^{-1}$  is  $d\alpha$  measurable. Consequently by

Theorem 4.2.14

$$\lim_{n \rightarrow \infty} \int_{\Delta} \alpha'(t) \sqrt{1-t^2} p_n^2(d\alpha, t) dt = \frac{1}{\pi} \int_{\Delta} \alpha'(t) dt$$

for every  $\Delta \subset (-1, 1)$ . By (22) we obtain

$$\int_{\Delta} d\alpha(t) = \int_{\Delta} \alpha'(t) dt,$$

that is  $\alpha_S(t) \equiv 0$  for  $-1 < t < 1$ . Finally we apply Theorem 3.3.7. If

$\sum_{j=0}^{\infty} j c_j^{0,1}(d\alpha) < \infty$  then we use Lemma 38 and Theorem 3.1.15.

Remark 41. In general the function  $\phi(\theta)$  in (20) does not coincide with

$\Gamma(\theta) + \theta - \frac{\pi}{2}$  where  $\Gamma$  is defined in Definition 4.2.4. For instance if  $\beta$

is the weight introduced in Theorem 22 then  $\phi(\theta) \neq \Gamma(\theta) + \theta - \frac{\pi}{2}$ . If we

know that  $\text{supp}(d\alpha) = [-1, 1]$  then by Theorem 1  $\alpha \in S$  and by Theorem 20

$\phi(\theta) = \Gamma(\theta) + \theta - \frac{\pi}{2}$ . Thus by Theorem 41 (14) holds uniformly for  $x \in \tau \subset$

$\subset (-1, 1)$  if the conditions of Theorem 20 are satisfied.

Theorem 42. Let  $\alpha \in M(0, 1)$  and  $\sum_{j=0}^{\infty} j c_j^{0,1}(d\alpha) < \infty$ . Then

$$\lim_{n \rightarrow \infty} p_n(d\alpha, z) \mu(z)^{-n-1} = \frac{1}{2\sqrt{z^2-1}} \phi(d\alpha, \mu(z)^{-1})$$

uniformly for  $|\mu(z)| \geq R > 1$  where  $\phi$  is defined by (19).  $\phi(d\alpha, \mu(z)^{-1})$  is

analytic in the domain  $|\mu(z)| > 1$  and vanishes for  $z \in \text{supp}(d\alpha) \setminus [-1, 1]$ .

Proof. Use Lemmas 38, 39 and Theorem 3.1.15. If  $x \in \text{supp}(d\alpha) \setminus [-1, 1]$

then by Theorem 3.1.7  $\alpha$  has a jump at  $x$ . Hence  $\lim_{n \rightarrow \infty} p_n(d\alpha, x) = 0$  that

is  $\phi(d\alpha, \mu(x)^{-1}) = 0$ .

Remark 43. If  $\text{supp}(d\alpha) = [-1, 1]$  then  $\alpha \in S$  and by Lemma 6.1.18

$$\frac{\mu(z)}{2\sqrt{z^2-1}} \phi(d\alpha, \mu(z)^{-1}) = \frac{1}{\sqrt{2\pi}} \text{Div} d\alpha, \mu(z)^{-1} - 1$$

for  $|\mu(z)| > 1$ .

Theorem 44. Let  $a \in M(0,1)$ ,  $\sum_{j=0}^{\infty} j c_j^{0,1}(da) < \infty$ . Let  $g(\geq 0)$  be Riemann integrable on  $[-1,1]$  and let  $g^{\pm 1}$  be bounded on  $\text{supp}(da)$ . Then

$$\lim_{n \rightarrow \infty} 2^{-n} Y_n(da_g) = a(da, 0) D(g, 0)^{-1},$$

$$\lim_{n \rightarrow \infty} P_n(da_g, z) \rho(z)^{-n-1} = \frac{1}{2\sqrt{z-1}} a(da, \rho(z)^{-1}) \cdot D(g, \rho(z)^{-1})^{-1}$$

for  $|\rho(z)| > 1$  and

$$\lim_{n \rightarrow \infty} \lambda_n(da_g, z)^{-1} |\rho(z)|^{-2n-2} =$$

$$= \frac{1}{4(|\rho(z)|^2 - 1)|z - 1|} |a(da, \rho(z)^{-1})|^2 |D(g, \rho(z)^{-1})|^{-2}$$

for  $|\rho(z)| > 1$ .

Proof. Use Theorems 42, 4.2.11, 6.1.25, 6.1.26 and 6.1.29.

Let us note that some results of Case [4] follows from the previous theorems. For example, Case proved Theorem 40 under the condition  $c_j^{0,1}(da) = O(j^{-2})$ .

## 8. Fourier Series

Lemma 1. Let  $\text{supp}(da)$  be compact,  $g \geq 0$ ,  $g^{\pm 1} \in L_{da}^1$ . Let  $f \in L_{da_g}^2$ . Then

$$(1) \quad |S_n(da_g, f, x) - \lambda_n(da, x) \lambda_n^{-1}(da_g, x) S_n(da, fg, x)| \leq$$

$$\leq \|f\|_{da_g, 2} \{ \lambda_n^{-1}(da_g, x) [G_n(da_g, g^{-1}, x) G_n(da, g, x) - 1] \}^{\frac{1}{2}}$$

for  $x \in \mathbb{R}$  and  $n = 1, 2, \dots$ .

Proof. Let us denote the left side in (1) by  $R(x)$ . Then

$$R(x) = \int_{-\infty}^{\infty} f(t) g(t) [K_n(da_g, x, t) - \frac{\lambda_n(da, x)}{\lambda_n(da_g, x)} K_n(da, x, t)] d\alpha(t).$$

Hence

$$|R(x)|^2 \leq \|f\|_{da_g, 2}^2 \cdot K(x)$$

where

$$K(x) = \int_{-\infty}^{\infty} [K_n(da_g, x, t) - \frac{\lambda_n(da, x)}{\lambda_n(da_g, x)} K_n(da, x, t)]^2 d\alpha_g(t).$$

Let us calculate  $K(x)$ . We have

$$K(x) = K_n(da_g, x, x) -$$

$$- 2 \frac{\lambda_n(da, x)}{\lambda_n(da_g, x)} \int_{-\infty}^{\infty} K_n(da_g, x, t) K_n(da, x, t) d\alpha_g(t) +$$

$$+ \frac{\lambda_n^2(da, x)}{\lambda_n^2(da_g, x)} \int_{-\infty}^{\infty} K_n^2(da, x, t) d\alpha_g(t) =$$

$$= \lambda_n^{-1}(da_g, x) \left[ \frac{\lambda_n(da, x)}{\lambda_n(da_g, x)} G_n(da, g, x) - 1 \right].$$

Now use Theorem 6.2.3.

Note that putting  $f = p_{n-1}(da_g)$  in (1) we obtain an inequality which may help us derive asymptotics for  $P_{n-1}(da_g, x)$ .

Recall that  $a_\tau, A_X^w, B_X^w, g$  etc. have been defined in 6.2.

**Theorem 2.** Let  $\alpha \in S$ ,  $f \in L^2_{da}$ . Let  $x \in (-1, 1)$ ,  $\alpha$  be absolutely continuous near  $x$ . Let  $\alpha' \in B^w_x$  with  $\omega(t)/t \in L^1$ ,  $\alpha'(x) > 0$ . If  $\tau^0$  is sufficiently small neighborhood of  $x$  then

$$(2) \quad \lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(da_\tau, f, l_\delta, x)] = 0$$

where  $l_\delta$  denotes the characteristic function of an arbitrary but fixed neighborhood of  $x$ . If  $\tau_1 \subset (-1, 1)$ ,  $\alpha$  is absolutely continuous in  $\tau_1(\epsilon)$ ,  $\alpha' \in B^w_{\tau_1}$  with  $\omega(t)/t \in L^1$ ,  $\alpha'(x) > 0$  for  $x \in \tau_1$  and  $\tau^0$  is a sufficiently small neighborhood of  $\tau_1$  then (2) holds uniformly for  $x \in \tau_1$  if  $l_\delta$  is the characteristic function of a neighborhood of  $\tau_1$ .

**Proof.** Since  $\alpha = (\alpha_\tau)_g$  we obtain from Theorems 6.2.40, 6.2.43, Remark 6.2.41 and Lemma 1 that

$$(3) \quad |S_n(da, f, x) - \frac{\lambda_n(da_\tau, x)}{\lambda_n(da, x)} S_n(da_\tau, fg, x)| \leq C \|f\|_{da, 2}$$

for  $n = 1, 2, \dots$ . Note that  $g$  is bounded, thus  $fg \in L^2_{da_\tau}$ . Let us consider now  $S_n(da_\tau, fg, x)$ . We have

$$(4) \quad S_n(da_\tau, fg, x) - g(x) S_n(da_\tau, f, x) = \int_{-1}^1 \frac{g(t) - g(x)}{t - x} f(t) K_n(da_\tau, x, t) da_\tau(t).$$

Since the sequence  $\{P_n(da_\tau, x)\}$  is uniformly bounded for  $x \in \tau^0 \subset \tau^0$  (See Lemma 6.2.29.) and

$$\int_{\tau_1} \int_{\tau_1} \frac{g(t) - g(x)}{t - x} |f(t)|^2 dt < \infty$$

we obtain from Bessel's inequality that the right side in (4) tends to 0 when  $n \rightarrow \infty$ . Further by Theorem 6.2.43

$$S_n(da_\tau, f, x) = \frac{\lambda_n(da_\tau, x)}{\lambda_n(da, x)} S_n(da_\tau, f, x) + \sigma(l).$$

To  $\alpha_\tau$  we can apply Freud's localization principle (Freud, §IV.5.), by which

$$S_n(da_\tau, f, x) = S_n(da_\tau, f, l_\delta, x) + \sigma(l).$$

Hence by (3)

$$(5) \quad \limsup_{n \rightarrow \infty} \max_{t \in x} |S_n(da_\tau, f, t) - S_n(da_\tau, f, l_\delta, t)| \leq C \|f\|_{da, 2} \quad (\text{or } t \in \tau_1)$$

where  $C$  does not depend on  $f$ . Putting here  $f = P$  instead of  $f$  where  $P$  is a polynomial with  $\|f - P\|_{da, 2} < \epsilon$  ( $\epsilon > 0$ ) and again using Freud's localization principle for  $\alpha_\tau$  we obtain that the left side in (5) is not greater than  $C\epsilon$ . Now let  $\epsilon \rightarrow 0$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

$$\frac{\lambda_n(da_\tau, x)}{\lambda_n(da, x)} = \frac{1}{\alpha'(x)} + O\left(\frac{1}{n}\right).$$

Hence

$$\frac{\lambda_n(da_\tau, x)}{\lambda_n(da, x)} S_n(da_\tau, fg, x) = S_n(da_\tau, f, x) +$$

$$+ O\left(\frac{1}{n}\right) S_n(da_\tau, f, x) + O\left(\frac{1}{n}\right) + \sigma(l).$$

We have further

$$|S_n(da_\tau, f, x)| \leq \|f\|_{da_\tau, 2} \lambda_n^{-1/2}(da_\tau, x) = O(\sqrt{n})$$

since  $da_\tau(t) = dt$  for  $t \in \tau$ . Thus

$$\frac{\lambda_n(da_\tau, x)}{\lambda_n(da, x)} = S_n(da_\tau, f, x) + \sigma(l).$$

To  $\alpha_\tau$  we can apply Freud's localization principle (Freud, §IV.5.), by which

$$S_n(da_\tau, f, x) = S_n(da_\tau, f, l_\delta, x) + \sigma(l).$$

Hence by (3)

$$(5) \quad \limsup_{n \rightarrow \infty} \max_{t \in x} |S_n(da_\tau, f, t) - S_n(da_\tau, f, l_\delta, t)| \leq C \|f\|_{da, 2} \quad (\text{or } t \in \tau_1)$$

where  $C$  does not depend on  $f$ . Putting here  $f = P$  instead of  $f$  where  $P$  is a polynomial with  $\|f - P\|_{da, 2} < \epsilon$  ( $\epsilon > 0$ ) and again using Freud's localization principle for  $\alpha_\tau$  we obtain that the left side in (5) is not greater than  $C\epsilon$ . Now let  $\epsilon \rightarrow 0$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

**Theorem 3.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\tau \subset (-1, 1)$ ,  $\alpha$  be absolutely continuous on  $\tau$ ,  $\alpha'(t) = 1$  for  $t \in \tau$ ,  $\tau_1 \subset \tau^0$ . Suppose that there exists a polynomial  $\pi$  such that  $\pi^2/\alpha' \in L^1(-1, 1)$ . Let  $f \in L^2_{da}$  and let  $l_\delta$  be the characteristic function of a sufficiently small neighborhood of  $\tau_1$ . Then

$$\lim_{n \rightarrow \infty} [S_n(da, f, x) - S_n(v, f, l_\delta, x)] = 0$$

uniformly for  $x \in \tau_1$ .

Proof. We could repeat Freud (§V.7.), but his proof can be simplified. He requires, moreover, that  $\pi^2/\alpha' \in L^\infty$ . First of all  $f|_{I_\delta} \in L^2_V$  for  $\delta$  small. Further,  $v^{-1}f|_{I_\delta} \in L^2_V$  also and it is easy to see that

$$\lim_{n \rightarrow \infty} [S_n(v, f|_{I_\delta}, x) - v(x) S_n(v, v^{-1}f|_{I_\delta}, x)] = 0$$

uniformly for  $x \in T_1$  since  $v$  is nice on  $T_1$ . By Lemma 6.2.29 and by

Freud's localization principle

$$\lim_{n \rightarrow \infty} [S_n(d\alpha, f, x) - S_n(d\alpha, f|_{I_\delta}, x)] = 0$$

uniformly for  $x \in T_1$ . By Theorem 6.2.46

$$\frac{\lambda_n(v, x)}{\lambda_n(d\alpha, x)} = v(x) + O\left(\frac{1}{n}\right) \quad (x \in T_1).$$

Hence

$$\frac{\lambda_n(v, x)}{\lambda_n(d\alpha, x)} S_n(v, v^{-1}f|_{I_\delta}, x) = v(x) S_n(v, v^{-1}f|_{I_\delta}, x) + O\left(\frac{1}{n}\right) \|v^{-1}f|_{I_\delta}\|_{V, 2}$$

( $x \in T_1$ ). Consequently we have to show that

$$|S_n(d\alpha, f|_{I_\delta}, x) - \frac{\lambda_n(v, x)}{\lambda_n(d\alpha, x)} S_n(v, v^{-1}f|_{I_\delta}, x)|$$

converges to 0 uniformly for  $x \in T_1$  when  $n \rightarrow \infty$ . But this expression

equals

$$\begin{aligned} & \left| \int_{T_1, \delta} f(t) [K_n(d\alpha, x, t) - \frac{\lambda_n(v, x)}{\lambda_n(d\alpha, x)} K_n(v, x, t)] dt \right| \leq \\ & \leq \|f|_{I_\delta}\|_{d\alpha, 2} \left\{ \lambda_n^{-1}(d\alpha, x) \int_{-1}^1 K_n^2(v, x, t) d\alpha(t) - 1 \right\}^{\frac{1}{2}}. \end{aligned}$$

By Theorem 6.2.46 and Lemma 6.2.44 the latter expression is not greater

than

$$C \|f|_{I_\delta}\|_{d\alpha, 2} \sqrt{n} \left\{ (v(x) + O\left(\frac{1}{n}\right))(v(x)^{-1} + O\left(\frac{1}{n}\right)) - 1 \right\}^{\frac{1}{2}} = O(1) \|f|_{I_\delta}\|_{d\alpha, 2}$$

and this is enough for our purposes.

From Theorems 2 and 3 we obtain the following equiconvergence

Theorem 4. Let  $\pi^2/\alpha' \in L^1(-1, 1)$  with a suitable polynomial  $\pi$ . If the conditions of the first part of Theorem 2 are satisfied then

$$\lim_{n \rightarrow \infty} [S_n(d\alpha, f, x) - S_n(v, f|_{I_\delta}, x)] = 0$$

and in the conditions of the second part of Theorem 2 the convergence is uniform for  $x \in T_1$ . Here  $I_\delta = [x - \delta, x + \delta]$  (or  $= I_{T_1}(\delta)$ ) and  $\delta$  is sufficiently small.

Corollary 5. Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\pi^2/\alpha' \in L^1$  with a suitable polynomial

$\pi$ . Let  $\alpha$  be absolutely continuous in  $\tau \subset (-1, 1)$ ,  $\alpha'(t) > C_1 > 0$  for  $t \in \tau$ ,  $\alpha' \in C^1(\tau)$ ,  $\omega(\alpha', t)/t \in L^1$  where  $\omega$  is the modulus of continuity of  $\alpha'$ .

Let  $f \in L^2_{d\alpha}$ . Then

$$\lim_{n \rightarrow \infty} S_n(d\alpha, f, x) = f(x)$$

for almost every  $x \in \tau$ .

Proof. Use Theorem 4 and Carleson [3].

In the following we shall investigate the Lebesgue functions

$$K_n(d\alpha, x) = \int_{-\infty}^{\infty} |K_n(d\alpha, x, t)| d\alpha(t)$$

$n = 1, 2, \dots$ . One trivial thing is sure:

$$K_n^2(d\alpha, x) \leq \lambda_n^{-1}(d\alpha, x) [\alpha(x) - \alpha(-\infty)].$$

Hence estimating  $\lambda_n^{-1}$  we obtain estimates for  $K_n$ . If e.g.  $\alpha'(t) > C > 0$  for  $t \in [x - \varepsilon, x + \varepsilon]$  then

$$K_n^2(d\alpha, x) \leq Cn.$$

It is rather surprising that nobody has tried to improve this estimate for

weights satisfying weak conditions (e.g. for  $\alpha \in S$ ). We shall see that

$Cn$  can be replaced by  $\sigma(n)$  in many cases. First we shall find conditions for



for  $x \in \Delta$ ,  $n = 1, 2, \dots$  where  $C = C(d\alpha, \Delta)$ .

Proof. Lemmas 6 and 3.3.1.

Theorem 8. Let  $\alpha \in M(0, 1)$ . If  $\alpha$  is continuous at  $x \in [-1, 1]$  then

$$(7) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) K_n^2(d\alpha, x) = 0.$$

If  $\alpha$  is continuous on the closed set  $\mathbb{M} \subset (-1, 1)$  then (7) is satisfied uniformly for  $x \in \mathbb{M}$ .

Proof. Use Corollary 7 and Theorem 4.1.11.

Theorem 9. Let  $\alpha \in M(0, 1)$ ,  $\tau \subset [-1, 1]$ ,  $\varepsilon > 0$ . If  $[\alpha']^{-\varepsilon} \in L^1(\tau)$  then

$$(8) \quad \lim_{n \rightarrow \infty} n^{-\frac{1}{2}} K_n(d\alpha, x) = 0$$

for almost every  $x \in \tau$ . If  $\alpha$  is continuous on  $\tau$  and  $\alpha'(t) \geq C > 0$  for almost every  $t \in \tau$  then (8) holds uniformly for  $x \in \tau \subset \tau^0$ .

Proof. Since  $\alpha$  is almost everywhere continuous in  $[-1, 1]$  the first part of the Theorem follows from Theorems 8 and 6.3.35. The second part follows from Theorem 8 and Example 6.2.9.

From Theorem 9 one can easily obtain convergence theorems for  $Lip_{\frac{1}{2}}^1$  (and not for  $lip_{\frac{1}{2}}^1$ ). Let us leave the details to the reader.

Let us note that (8) is good for bad weights, for nice weights better estimates can be found.

Theorem 10. Let  $\text{supp}(d\alpha)$  be compact,  $\tau \subset \text{supp}(d\alpha)$ ,  $\varepsilon > 0$ ,  $[\alpha']^{-\varepsilon} \in L^1(\tau)$ .

Then

$$\liminf_{n \rightarrow \infty} n^{-\frac{1}{3}} K_n(d\alpha, x) < \infty$$

for almost every  $x \in \tau$ .

Proof. Let us put in Lemma 6  $\varepsilon = n^{-\frac{1}{3}}$ . Using Lemma 3.3.1 we obtain

$$(6) \quad \lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) K_n^2(d\alpha, x) = 0.$$

If  $\text{supp}(d\alpha)$  is compact and  $\alpha$  has a jump at  $x$  then

$$\liminf_{n \rightarrow \infty} \lambda_n(d\alpha, x) K_n^2(d\alpha, x) \geq \alpha(x+0) - \alpha(x-0)$$

so that (6) cannot hold.

Lemma 6. Let  $\varepsilon > 0$ ,  $x \in \mathbb{R}$ . Then

$$\begin{aligned} \lambda_n(d\alpha, x) K_n^2(d\alpha, x) &\leq 2[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] + \\ &+ \frac{2}{\varepsilon^2} \lambda_n(d\alpha, x) \left[ \frac{Y_{n-1}^2(d\alpha)}{Y_n^2(d\alpha)} + 2(x - \alpha_{n-1}(d\alpha))^2 \right] p_{n-1}^2(d\alpha, x) + \\ &+ 2 \frac{Y_{n-2}^2(d\alpha)}{Y_{n-1}^2(d\alpha)} p_{n-2}^2(d\alpha, x) [\alpha(\infty) - \alpha(-\infty)]. \end{aligned}$$

Proof. We shall use the Christoffel-Darboux and the recurrence formulas.

We have

$$K_n(d\alpha, x) = \left\{ \int_{|x-t| \leq \varepsilon} \right\} + \left\{ \int_{|x-t| > \varepsilon} \right\} |K_n(d\alpha, x, t)| d\alpha(t).$$

Hence

$$K_n^2(d\alpha, x) \leq 2 \left\{ \int_{|x-t| \leq \varepsilon} \right\}^2 + 2 \left\{ \int_{|x-t| > \varepsilon} \right\}^2.$$

Here the first integral in the square is not greater than

$$[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t) d\alpha(t) = \lambda_n^{-1}(d\alpha, x) [\alpha(x+\varepsilon) - \alpha(x-\varepsilon)].$$

Further

$$\left\{ \int_{|x-t| \geq \varepsilon} \right\}^2 \leq \frac{1}{\varepsilon^2} \frac{Y_{n-1}^2(d\alpha)}{Y_n^2(d\alpha)} [p_{n-1}^2(d\alpha, x) + p_n^2(d\alpha, x)] \cdot [\alpha(\infty) - \alpha(-\infty)].$$

Now the final estimate follows from the recurrence formula.

Corollary 7. Let  $\text{supp}(d\alpha)$  be compact,  $\varepsilon > 0$  and  $\Delta$  be fixed. Then

$$\begin{aligned} \lambda_n(d\alpha, x) K_n^2(d\alpha, x) &\leq 2[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] + \\ &+ C \varepsilon^{-2} \lambda_n(d\alpha, x) [p_{n-1}^2(d\alpha, x) + p_n^2(d\alpha, x)] \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{n} \frac{1}{3} K_n^2(d\alpha, x) \leq 2[n \lambda_n(d\alpha, x)]^{-1} \cdot \\
& \cdot \frac{1}{3} \left[ \frac{1}{\alpha(x+n)} - \frac{1}{\alpha(x-n)} - \frac{1}{3} \right] + \\
& + C p_{n-1}^2(d\alpha, x) + C p_{n-2}^2(d\alpha, x).
\end{aligned}$$

Summing for  $n = 1, 2, \dots, m$  we see that

$$\begin{aligned}
& \frac{1}{m} \sum_{n=1}^m \frac{1}{n} \frac{2}{3} K_n^2(d\alpha, x) \leq \\
& \leq 2 \frac{1}{m} \sum_{n=1}^m [n \lambda_n(d\alpha, x)]^{-1} \frac{1}{3} \left[ \frac{1}{\alpha(x+n)} - \frac{1}{\alpha(x-n)} - \frac{1}{3} \right] + C[m \lambda_m(d\alpha, x)]^{-1}.
\end{aligned}$$

Since  $\alpha$  is almost everywhere differentiable we obtain from Theorem 6.3.35 that

$$\limsup_{n \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \frac{1}{n} \frac{2}{3} K_n^2(d\alpha, x) < \infty$$

for almost every  $x \in \tau$ . Hence the Theorem follows.

Now we shall be again in the situation  $\alpha, \tau \rightarrow g, \alpha_\tau$ . (See 6.2.)

**Lemma 11.** Let  $\alpha \in S, \tau \subset (-1, 1)$ . Then

$$K_n(d\alpha_\tau, x) \leq C \log n \quad (n \geq 3)$$

uniformly for  $x \in \tau_1 \subset \tau^0$ .

**Proof.** Lemma 6.2.29 and Freud, §IV, 4.

**Theorem 12.** Let  $\alpha \in S$ . Let  $x \in (-1, 1)$ ,  $\alpha$  be absolutely continuous near  $x$ ,  $\alpha' \in A_x^\omega, \alpha'(x) > 0$ . Then

$$(9) \quad K_n(d\alpha, x) \leq C \left\{ \log n + \left[ \int_1^n \frac{\omega(t)}{t^2} dt \right]^{\frac{1}{2}} \right\} \quad (n \geq 3)$$

where  $C$  does not depend on  $n$ . If  $\tau_1 \subset (-1, 1)$ ,  $\alpha$  is absolutely continuous near  $\tau_1$ ,  $\alpha' \in A_{\tau_1}^\omega, \alpha'(t) > 0$  for  $t \in \tau_1$  then (9) holds uniformly for  $x \in \tau_1$  with  $C$  independent of  $x$  and  $n$ .

**Proof.** Let us choose  $\tau$  ( $x \in \tau^0$  or  $\tau_1 \subset \tau^0$ ) so small that the corresponding  $g$  is bounded from below and above in  $[-1, 1]$ . We have  $\alpha = (\alpha_\tau)_g$ . Hence by Lemma 1

$$\begin{aligned}
K_n(d\alpha, x) & \leq C \lambda_n(d\alpha_\tau, x) \lambda_n^{-1}(d\alpha, x) K_n(d\alpha_\tau, x) + \\
& + C \{ \lambda_n^{-1}(d\alpha, x) [G_n(d\alpha_\tau, g^{-1}, x) G_n(d\alpha_\tau, g, x) - 1] \}^{\frac{1}{2}}.
\end{aligned}$$

By Theorem 6.2.6

$$\lambda_n(d\alpha_\tau, x) \lambda_n^{-1}(d\alpha, x) \leq C.$$

By Example 6.2.9  $\lambda_n^{-1}(d\alpha, x) \leq Cn$ . Now the Theorem follows from Lemma 11, Theorem 6.2.38, Remark 6.2.41 and Lemma 6.2.29.

Note that if  $\omega(t) = t |\log t|$  then

$$\int_1^n \frac{\omega(t)}{t^2} dt \sim [\log n]^2.$$

A weaker version of Theorem 12 was obtained by Freud, §V.7.

**Theorem 13.** Let  $\alpha \in S, u(\geq 0) \in L_{d\alpha}^1, w(\geq 0) \in L_{d\alpha}^1, \text{meas}(u > 0) > 0$ ,  $\text{meas}(w > 0) > 0, 1 < q \leq \infty, u^{1/1-q} \in L_{d\alpha}^1 (q < \infty), u^{-1} \in L_{d\alpha}^1 (q = \infty), 0 < p < \infty, p \leq q$ . If  $q < \infty$  and for every  $f \in L_{ud\alpha}^1$

$$\|S_n(d\alpha, f)\|_{wd\alpha, p} \leq C \|f\|_{ud\alpha, q}$$

for  $n = 1, 2, \dots$  with  $C$  independent of  $n$  and  $f$  then

$$(10) \quad \int_{-1}^1 \left[ \alpha'(t) \sqrt{1-t^2} \right]^{\frac{p}{2}} w(t) \alpha'(t) dt < \infty$$

and

$$\int_{-1}^1 \left[ \alpha'(t) \sqrt{1-t^2} \right]^{\frac{q}{2(1-q)}} u(t)^{1-q} \alpha'(t) dt < \infty.$$

If  $q = \infty$  and for  $u \in L_{dq}^1$

$$\|S_n(d\alpha, f)\|_{wd\alpha, p} \leq C \|u f\|_{d\alpha, q}$$

## 9. Inequalities

When investigating the Lebesgue functions of Lagrange interpolating processes we shall have to be able to estimate the expression

$$(1) \quad \sum_{|x-x_{kn}| < \varepsilon} \lambda_{kn}(d\alpha).$$

The following result is very simple.

Lemma 1. Let  $\text{supp}(d\alpha)$  be compact. Then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \sum_{|x-x_{kn}| < \varepsilon} \lambda_{kn}(d\alpha) = \alpha(x+0) - \alpha(x-0)$$

for each  $x \in \mathbb{R}$ .

Proof. For every  $\varepsilon > 0$  we can find  $y_1 \in (x-2\varepsilon, x-\varepsilon)$  and  $y_2 \in (x+\varepsilon, x+2\varepsilon)$  such that  $\alpha$  is continuous at  $y_1$  and  $y_2$ . Thus by using convergence

theorems for mechanical quadrature processes (See e.g. Freud, §III.1) we

obtain

$$\limsup_{n \rightarrow \infty} \sum_{|x-x_{kn}| < \varepsilon} \lambda_{kn}(d\alpha) \leq \int_{x-y_1}^{x+y_2} d\alpha(t) \leq \alpha(x+2\varepsilon) - \alpha(x-2\varepsilon).$$

Now let  $\varepsilon \rightarrow 0$ .

Unfortunately, it is not true that (1) is not greater than  $\alpha(x+\varepsilon) - \alpha(x-\varepsilon)$  or  $\alpha(x+2\varepsilon) - \alpha(x-2\varepsilon)$ . From the Markov-Stieltjes inequalities we obtain that

$$\sum_{|x-x_{kn}| < \varepsilon} \lambda_{kn}(d\alpha) \leq \alpha(x^1) - \alpha(x^2)$$

where  $x^1 = \min_{x_{kn} \geq x+\varepsilon} x_{kn}$  if  $\{k : x_{kn} \geq x+\varepsilon\}$  is not empty and otherwise  $x^1 = +\infty$  and similarly  $x^2 = \max_{x_{kn} \leq x-\varepsilon} x_{kn}$  or  $x^2 = -\infty$ . Suppose that neither

$\{k : x_{kn} < x^1\}$  nor  $\{k : x_{kn} > x^2\}$  is empty. Let  $x_1^1 = \max_{x_{kn} < x^1} x_{kn}$  and

$x_2^2 = \min_{x_{kn} > x^2} x_{kn}$ . Then

with  $C \neq C(n, f)$  for  $n = 1, 2, \dots$  then (10) and

$$\int_{-1}^1 |\alpha'(t)| \sqrt{1-t^2}^{-\frac{1}{2}} u(t)^{-1} \alpha'(t) dt < \infty$$

hold.

Proof. For simplicity we shall consider the case  $1 < q < \infty$ . By the conditions

$$\|S_n(d\alpha, f) - S_{n-1}(d\alpha, f)\|_{w d\alpha, p} \leq C \|f\|_{u d\alpha, q}.$$

This means that

$$\|p_n(d\alpha)\|_{w d\alpha, p} \cdot \|f p_n(d\alpha)\|_{d\alpha, 1} \leq C \|f\|_{u d\alpha, q}.$$

By Hölder's inequality this is equivalent to

$$\sup_{n \geq 1} (\|p_n(d\alpha)\|_{w d\alpha, p} \cdot \|p_n(d\alpha)u^{-1}\|_{u d\alpha, q'}) < \infty$$

where  $q' = q/(q-1)$ . By Theorem 4.2.8 the latter condition is equivalent to

$$\sup_{n \geq 1} \|p_n(d\alpha)\|_{w d\alpha, p} < \infty$$

and

$$\sup_{n \geq 1} \|p_n(d\alpha)u^{-1}\|_{u d\alpha, q'} < \infty.$$

The Theorem follows now from Theorems 7.31 and 7.32.

Let us note that many special cases of Theorem 13 have been known.

We refer to Badkov [2] and to the literature mentioned there. (In particular, to Askey, Muckenhaupt, Newman-Rudin, Pollard, Stein and Wainger.)

Corollary 14. There exists an absolutely continuous  $\alpha \in S$  such that from

$$(11) \quad \sup_{n \geq 1} \|S_n(d\alpha)\|_{L_{d\alpha}^p} \rightarrow L_{d\alpha}^p < \infty$$

follows  $p = 2$ .

Proof. Put  $\alpha'(x) = \exp\{-(1-x^2)^{-\frac{1}{2}}\}$ . If  $1 < p < \infty$  then apply Theorem 13.

For  $p = 1, \infty$  (11) can never hold.



$$(2) \quad \sum_{|x-x_{kn}| < \varepsilon} \lambda_{kn}(d\alpha) \leq \alpha(x+\varepsilon + x^1 - x^1_1) - \alpha(x-\varepsilon + x^2 - x^2_2).$$

Hence we see that to estimate (1) we have to know the behavior of  $x_k(d\alpha) - x_{k+1,n}(d\alpha)$ .

Lemma 2. Let  $\beta \in S$ . Then there exists a number  $C = C(d\beta) > 1$  such that

$$p_n^2(d\beta, x) \leq C \sqrt{n}$$

for  $x \in [-1, 1]$  and  $n = 1, 2, \dots$ .

Proof. See Geronimus, §8.2.

Lemma 3. Let  $\alpha$  be an arbitrary weight,  $\Delta \subset \text{supp}(d\alpha)$ . Let  $v_\Delta$  denote the Tschebyshev weight corresponding to  $\Delta$ . If  $v_\Delta \log \alpha' \in L^1(\Delta)$  then

$$\max_{x \in \Delta} \pi_n^2(x) \leq C \sqrt{n} \int_{-\infty}^{\infty} \pi_n^2(t) d\alpha(t) \quad (n \geq 1)$$

for each  $\pi_n$  with a suitable  $C = C(d\alpha) > 1$ .

Proof. Let  $\alpha^*(t) = \alpha(t)$  for  $t \in \Delta$  and  $\alpha^*(t)$  be constant otherwise. Let us transform  $\Delta$  into  $[-1, 1]$ . We get a weight  $\alpha^{**}$  which satisfies the conditions of Lemma 2. Returning to  $\Delta$  we obtain

$$\begin{aligned} \max_{x \in \Delta} \pi_n^2(x) &\leq \max_{x \in \Delta} \lambda_{n+1}^{-1}(d\alpha^*, x) \int_{\Delta} \pi_n^2(t) d\alpha^*(t) \leq \\ &\leq n C \sqrt{n} \int_{\Delta} \pi_n^2(t) d\alpha(t) \leq C \sqrt{n} \int_{-\infty}^{\infty} \pi_n^2(t) d\alpha(t). \end{aligned}$$

Theorem 4. Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $v_\Delta \log \alpha' \in L^1(\Delta)$ .

Then

$$(3) \quad x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \leq C \frac{1}{\sqrt{n}} \quad (n \geq 1)$$

for  $x_k, x_{k+1} \in \Delta$  with  $C$  independent of  $n$  and  $k$ . If  $\Delta_1 \subset \Delta^0$  then (3) holds if either  $x_k$  or  $x_{k+1}$  belongs to  $\Delta_1$ .

Proof. Let  $v^*$  denote the Tschebyshev weight corresponding to  $\Delta(d\alpha)$ .

Let  $m$  be a natural integer and  $N = \lfloor \frac{n}{m} \rfloor$ . Then

$$\pi_x(t) = K_N^m(v^*, x, t) K_N^{*-m}(v^*, x, x)$$

is a  $\pi_{n-1}$  with  $\pi_x(x) = 1$ . By Lemma 3

$$(4) \quad 1 \leq C \sqrt{n} \int_{-\infty}^{\infty} \pi_x^2(t) d\alpha(t)$$

for  $x \in \Delta$ . Let  $x_k, x_{k+1} \in \Delta$  and  $x = \frac{1}{2}(x_k + x_{k+1})$ . Then  $x \in \Delta$ . Further

$$|\pi_x(x_j n(d\alpha))| \leq [C \frac{m}{n} (x_k - x_{k+1})^{-1}]^m$$

for  $j = 1, 2, \dots, n$  with  $C_1 = C_1(\Delta(d\alpha))$ . Calculating the integral on the right

side of (4) by the Gauss-Jacobi mechanical quadrature formula we obtain

$$[x_k - x_{k+1}]^{2m} \leq C \sqrt{n} [C_1 \frac{m}{n}]^{2m} [\alpha(\infty) - \alpha(-\infty)],$$

that is

$$x_k - x_{k+1} \leq C_2 \frac{\sqrt{n}}{C^{2m} \frac{m}{n}}.$$

Putting here  $m = \lfloor \sqrt{n} \rfloor$  the first part of the Theorem follows. Using Lemma

3.3.2 we obtain the second part of the Theorem.

Theorem 5. Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $\varepsilon > 0$ ,  $[\alpha']^{-\varepsilon} \in L^1(\Delta)$

Then

$$(5) \quad x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \leq C \frac{\log n}{n} \quad (n \geq 3)$$

for  $x_k, x_{k+1} \in \Delta$  where  $C$  does not depend on  $n$  and  $k$ . If  $\Delta_1 \subset \Delta^0$  then

$$(5) \text{ holds for either } x_k \in \Delta_1 \text{ or } x_{k+1} \in \Delta_1.$$

Proof. We obtain from Theorem 6.3.13 and from  $[\alpha']^{-\varepsilon} \in L^1(\Delta)$  that

$$\max_{x \in \Delta} \pi_n^2(x) \leq n^A \int_{-\infty}^{\infty} \pi_n^2(t) d\alpha(t) \quad (n \geq 2)$$

for every  $\pi_n$  with a suitable constant  $A > 1$ . Now we repeat the proof of

Theorem 4 and finally we put  $m = \lfloor \log n \rfloor$ .



Let us note that the proof of Theorems 4 and 5 is based on an idea of Erdős-Turán but our result is stronger than that of Erdős-Turán. (See Szegő, §6.11.)

**Theorem 6.**

Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $\forall \log x \in L(\Delta)$ ,  $\epsilon_1 > 0$ ,  $\tau \in \tau(\epsilon_1) \subset \Delta^0$ . Then

$$\sum_{|x-x_{kn}| < \epsilon} \lambda_{kn}(d\alpha) \leq \alpha(x+\frac{C}{\sqrt{n}}) - \alpha(x-\frac{C}{\sqrt{n}})$$

uniformly for  $n=1, 2, \dots$ ,  $x \in \tau$ ,  $0 \leq \epsilon \leq \epsilon_1$  where  $C = C(n, x, \epsilon)$ ,  $C > 0$ .

**Proof.** Use (2) and Theorem 4.

**Lemma 7.** Let  $\text{supp}(d\alpha)$  be compact,  $\epsilon > 0$ ,  $\tau \in \tau(\epsilon) \subset \Delta^0 \subset \Delta \subset \text{supp}(d\alpha)$ .

Then there exists a number  $N = N(\epsilon, d\alpha, \Delta)$  such that

$$\sum_{|x-x_{kn}| < \epsilon} \lambda_{kn}(d\alpha) \leq \alpha(x+2\epsilon) - \alpha(x-2\epsilon)$$

for  $x \in \tau$  and  $n \geq N$ .

**Proof.** Apply Lemma 3.2.2, (2) and the Heine-Borel theorem.

In the following we shall also need estimates for

$$(6) \quad \sum_{x_{kn} \in \tau} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})|.$$

It is obvious that (6) is not greater than  $[\alpha(\infty) - \alpha(-\infty)]^{\frac{1}{2}}$ . The question if (6) may converge to 0 when  $n \rightarrow \infty$  seems to be more difficult.

**Example 8.** Let  $w$  be the Hermite weight, that is  $w(x) = \exp(-x^2)$  for  $x \in \mathbb{R}$ .

Then

$$|p_{n-1}(w, x_{kn})| \leq C n^{-\frac{1}{4}} w(x_{kn})^{-\frac{1}{2}}$$

for  $k = 1, 2, \dots, n$ . Hence by old theorems about quadrature sums

$$\sum_{k=1}^n \lambda_{kn}(w) |p_{n-1}(w, x_{kn})| \leq C n^{-\frac{1}{4}} \int_{-\infty}^{\infty} w(t)^{\frac{1}{2}} dt \xrightarrow{n \rightarrow \infty} 0.$$

We shall show that this cannot happen if  $\text{supp}(d\alpha)$  is compact and  $\alpha$  is nice in a certain sense. For  $D(d\alpha, 0)$  see Definition 6.1.16.

**Lemma 9.** Let  $\alpha \in S$ . Then

$$\liminf_{n \rightarrow \infty} \sum_{k=1}^n \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| \geq \frac{\sqrt{\pi}}{2} D(d\alpha, 0).$$

**Proof.** Let  $n \geq 1$ . Then  $T_{n-1}(x) = L_n(d\alpha, T_{n-1}, x)$ . Let us divide both sides by  $x^{n-1}$  and let  $x \rightarrow \infty$ . We obtain

$$2^{n-2} \leq \gamma_{n-1}(d\alpha) \sum_{k=1}^n \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})|.$$

Now apply Lemma 4.2.2.

The following result is a poor but very useful analogue of Theorem

4.2.8.

**Theorem 10.** Let  $\alpha \in S$ . Then there exists a number  $\delta = \delta(d\alpha) > 0$  such that if  $\Omega \subset [-1, 1]$  is an arbitrary finite system of disjoint intervals with  $|\Omega| \geq 2-\delta$

then

$$(7) \quad \liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| > 0.$$

**Proof.** Let  $\alpha\Omega = [-1, 1] \setminus \Omega$ . Then  $L_{\alpha\Omega}$  is Riemann integrable on  $[-1, 1]$ . We have

$$\sum_{k=1}^n \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| = \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| +$$

$$+ \sum_{k=1}^n L_{\alpha\Omega}(x_{kn}) \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| \leq$$

$$\leq \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| +$$

$$+ [( \alpha(1) - \alpha(-1) ) \sum_{k=1}^n L_{\alpha\Omega}(x_{kn}) \lambda_{kn}(d\alpha) p_{n-1}^2(d\alpha, x_{kn})]^{\frac{1}{2}}.$$

Now let  $n \rightarrow \infty$ . By Lemma 9 and Theorem 3.2.3

$$\begin{aligned} \frac{\sqrt{n}}{2} D(d\alpha, 0) &\leq \liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| + \\ &+ [(\alpha(1) - \alpha(-1)) \frac{2}{\pi} \int_{c\Omega} \sqrt{1-t^2} dt]^{\frac{1}{2}}. \end{aligned}$$

Hence (7) holds if  $|c\Omega|$  is small.

Theorem 11. Let  $\alpha \in M(0, 1)$ ,  $\tau \in [-1, 1]$ . Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \liminf_{x_{kn} \in \tau} \sum_{x_{kn} \in \tau} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| &\geq \frac{2}{\pi} \int_{\tau} \sqrt{1-t^2} dt. \\ \cdot \left\{ \liminf_{n \rightarrow \infty} \max_{x_{kn} \in \tau} |p_{n-1}(d\alpha, x_{kn})| \right\}^{-1}. \end{aligned}$$

Proof. The Theorem follows from Theorem 3.2.3 and the inequality

$$\begin{aligned} \sum_{x_{kn} \in \tau} \lambda_{kn}(d\alpha) p_{n-1}^2(d\alpha, x_{kn}) &\leq \max_{x_{kn} \in \tau} |p_{n-1}(d\alpha, x_{kn})| \cdot \\ \cdot \sum_{x_{kn} \in \tau} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})|. \end{aligned}$$

Theorem 12. Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\alpha'(x) > 0$  for almost every  $x \in (-1, 1)$ ,  $\Delta \subset [-1, 1]$ . Then

$$\begin{aligned} (8) \quad \liminf_{n \rightarrow \infty} \left\{ \max_{x \in \Delta} |p_n(d\alpha, x)| \sum_{x_{kn} \in \Delta} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| \right\} &\geq \\ &\geq \frac{1}{2\pi} \int_{\Delta} v(t)^{-1} v(t) dt. \end{aligned}$$

Proof. By Bernstein's inequality

$$|p'_n(d\alpha, t)| \leq \frac{2n}{|\Delta|} v(t) \max_{x \in \Delta} |p_n(d\alpha, x)|$$

for  $x \in \Delta$ . Further  $\gamma_{n-1}(d\alpha) \leq \gamma_n(d\alpha)$  and

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha) p_{n-1}(d\alpha, x_{kn}) = [p'_n(d\alpha, x_{kn})]^{-1}.$$

Hence the left side in (8) is not less than

$$\frac{1}{2} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{x_{kn} \in \Delta} v(x_{kn})^{-1}.$$

But

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x_{kn} \in \Delta} v(x_{kn})^{-1} = \frac{1}{\pi} \int_{\Delta} v(t)^{-1} v(t) dt.$$

(See Freud, §III.9.)

Corollary 13. Let the conditions of Theorem 12 be satisfied. Let the sequence  $\{|p_n(d\alpha, x)|\}$  be uniformly bounded for  $x \in \Delta$ . Then

$$\liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Delta} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| > 0.$$

Let us note that the Pollaczek weight satisfies the conditions of Corollary 13.

Now we shall deal with weighted Bernstein-Markov inequalities. Our aim will be to generalize the following result of Khalilova [9].

Lemma 14. Let  $1 \leq p < \infty$ ,  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$  and  $u$  be a Jacobi weight. Then

$$\|\pi'_n v^{-1}\|_{u,p} \leq C n \|\pi\|_{u,p}$$

and

$$\begin{aligned} \max_{|x| \leq 1} \{ |\pi'_n(x)| (\sqrt{1-x} + \frac{1}{n})^{\frac{1}{a} + 1} (\sqrt{1+x} + \frac{1}{n})^{\frac{1}{b} + 1} \} &\leq \\ \leq C n \max_{|x| \leq 1} \{ |\pi_n(x)| (\sqrt{1-x} + \frac{1}{n})^a (\sqrt{1+x} + \frac{1}{n})^b \} \end{aligned}$$

for every  $\pi_n$  where  $C$  does not depend on  $\pi_n$  and  $n$ .

Lemma 15. Let  $a > -1$ ,  $1 \leq p \leq \infty$ . Then

$$(9) \quad \int_{-1}^1 |\pi'_n(t)|^p |t|^a dt \leq C n^p \int_{-2}^2 |\pi_n(t)|^p |t|^a dt$$

for every  $\pi_n$ .

Proof. Let first  $\pi_n$  be even, that is let  $\pi_n(x) = G_n(x^2)$ . Then  $\pi'_n(x) = 2x G'_n(x^2)$  and we have to show that

$$\int_{-1}^1 |x G'_n(x^2)|^p |x|^a dx \leq C n^p \int_{-2}^2 |G_n(x^2)|^p |x|^a dx$$

or

$$\int_0^1 |G'_n(x)|^p |x|^{\frac{p(a-1)}{2}} dx \leq C n^p \int_0^4 |G_n(x)|^p |x|^{\frac{p(a-1)}{2}} dx.$$

But

$$\int_0^1 |G'_n(x)|^p |x|^{\frac{p(a-1)}{2}} dx \leq C \int_0^4 |G'_n(x)|^p \sqrt{x} |x|^{\frac{p(a-1)}{2}} dx.$$

Hence (9) follows from Lemma 14 when  $\pi_n$  is even. Let now  $\pi_n$  be odd:

$\pi_n(x) = x G_n(x^2)$ . In this case  $\pi'_n(x) = G_n(x^2) + 2x^2 G'_n(x^2)$  and we shall prove that

$$\int_{-1}^1 |x^2 G'_n(x^2)|^p |x|^a dx \leq C n^p \int_{-2}^2 |x G_n(x^2)|^p |x|^a dx$$

and

$$\int_{-1}^1 |G_n(x^2)|^p |x|^a dx \leq C n^p \int_{-2}^2 |x G_n(x^2)|^p |x|^a dx.$$

The first inequality here follows from the first part of the proof by putting there

$p+a$  instead of  $a$ . The second inequality has been proved in Corollary 6.3.26.

Hence (9) holds if  $\pi_n$  is either even or odd. But then it also holds for every

$\pi_n$  with a possibly bigger constant  $C$ .

From Lemmas 14 and 15 follows

Theorem 16. Let  $1 = t_1 > t_2 > \dots > t_N = -1$ ,  $\Gamma_k > -1$  for  $k = 1, 2, \dots, N$ ,

$$w(t) \sim \prod_{k=1}^N |t - t_k|^{\Gamma_k} \quad (-1 \leq t \leq 1).$$

-205-

Let  $1 \leq p < \infty$ . Then for every  $\pi_n$

$$\|\pi'_n v^{-1}\|_{w,p} \leq C n \|\pi_n\|_{w,p}$$

where  $C$  does not depend on  $n$  and  $\pi_n$ .

Lemma 17. Let  $a \in \mathbb{R}$ . Then there exists a number  $\varepsilon = \varepsilon(a) > 0$  such that for every  $\pi_n$

$$(10) \quad \max_{|x| \leq \frac{\varepsilon}{n}} |\pi_n(x)| \leq C n^a \max_{\frac{\varepsilon}{n} \leq |x| \leq 1} \{ |x|^a |\pi_n(x)| \}$$

with  $C = C(a)$ .

Proof. Let first  $a = 0$ . Then (10) follows from Lemmas 6.2.50, 6.3.5 and 6.3.22 applied for the Legendre weight. If (10) holds for  $a = 0$  then it also

holds for  $a \geq 0$ . If  $a < 0$  then we remark that (10) with  $a = 0$  implies that

$$(11) \quad \max_{|x| \leq \frac{\varepsilon}{n}} |\pi_n(x)| \leq C \max_{\frac{\varepsilon}{n} \leq |x| \leq \frac{1}{n}} |\pi_n(x)|$$

with a possibly new  $\varepsilon > 0$  and  $C$ . Let  $w$  be defined by  $w(t) = |t|^{-a}$  for

$-1 \leq t \leq 1$ ,  $\text{supp}(w) = [-1, 1]$ . Putting in (11)  $K_n(w, x, x) \pi_n(x)$  instead of  $\pi_n(x)$

we obtain from Lemma 6.3.19

$$\max_{|x| \leq \frac{\varepsilon}{3n}} \frac{\varepsilon}{n} |\pi_n(x)| n^{1-a} \leq C \max_{\frac{1}{3n} \leq |x| \leq \frac{1}{n}} \{ |\pi_n(x)| |x|^a \} n.$$

Hence (1) follows for  $a < 0$  also.

Lemma 18. Let  $a \in \mathbb{R}$ . Then

$$\max_{|x| \leq 1} \{ |\pi'_n(x)| (|x| + \frac{1}{n})^a \} \leq C n \max_{|x| \leq 2} \{ |\pi_n(x)| (|x| + \frac{1}{n})^a \}$$

for every  $\pi_n$ .

Proof. Repeat the proof of Lemma 15 and use Lemmas 14 and 17.

By Lemmas 14 and 18 we obtain the following

Theorem 19. Let  $A_k \in \mathbb{R}$  for  $k = 1, 2, \dots, N$ ,  $1 > t_2 > \dots > t_{N-1} > -1$ ,

-206-

$$w_n(x) = (\sqrt{t-x} + \frac{1}{n})^{2A} \prod_{k=2}^{N-1} (|x-t_k| + \frac{1}{n})^{A_k} (\sqrt{t-x} + \frac{1}{n})^{2A_N}$$

for  $|x| \leq 1$ . Then

$$\max_{|x| \leq 1} \{ |\tau'_n(x)| w_n(x) (\sqrt{t-x} + \frac{1}{n}) (\sqrt{t-x} + \frac{1}{n}) \} \leq$$

$$< C n \max_{|x| \leq 1} \{ |\tau'_n(x)| w_n(x) \}$$

for every  $\tau_n$  where  $C$  is independent of  $n$  and  $\tau_n$ .

Now we return to estimating the distance between two consecutive zeros of orthogonal polynomials.

Theorem 20. Let  $\text{supp}(d\alpha)$  be compact,  $\Delta \subset \text{supp}(d\alpha)$ ,  $t \in \Delta^0$ ,  $\Gamma > -1$ .

Let  $\alpha$  be absolutely continuous in  $\Delta$  with

$$\alpha'(t) \sim |t-t|^\Gamma \quad \forall t \in \Delta.$$

Then

$$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \sim \frac{1}{n}$$

for  $x_{kn} \in \Delta_1 \subset \Delta^0$ .

Proof. By Lemma 3.3.2 we can suppose that both  $x_{kn}$  and  $x_{k+1,n}$  belong to  $\Delta_1$ . First we shall show that

$$x_{kn} - x_{k+1,n} \leq C n^{-1}.$$

We have the following possibilities.

$$(12) \quad t^* \leq x_{k+1} < x_k \leq t^* + \frac{1}{n} \quad \text{or} \quad t^* - \frac{1}{n} \leq x_{k+1} < x_k \leq t^*$$

$$\quad \text{or} \quad t^* - \frac{1}{n} < x_{k+1} \leq t^* < x_k < t^* + \frac{1}{n},$$

$$(13) \quad t^* \leq x_{k+1} \leq t^* + \frac{1}{n} < x_k \quad \text{or} \quad x_{k+1} < t^* - \frac{1}{n} \leq x_k \leq t^*,$$

$$(14) \quad t^* + \frac{1}{n} \leq x_{k+1} < x_k \quad \text{or} \quad x_{k+1} < x_k \leq t^* - \frac{1}{n},$$

$$(15) \quad x_{k+1} \leq t^* - \frac{1}{n} < t^* < x_k \leq t^* + \frac{1}{n} \quad \text{or}$$

$$t^* - \frac{1}{n} \leq x_{k+1} \leq t^* < t^* + \frac{1}{n} \leq x_k$$

and

$$(16) \quad x_{k+1} \leq t^* - \frac{1}{n} < t^* + \frac{1}{n} \leq x_k.$$

In all cases (12)-(16) we shall use Theorem 6.3.25 and the estimate

$$\int_{x_{k+1,n}}^{x_{kn}} d\alpha(t) \leq \lambda_{kn}(d\alpha) + \lambda_{k+1,n}(d\alpha) \quad (k = 1, 2, \dots, n-1)$$

which follows from the Markov-Stieltjes inequalities. In the first case of (13)

we obtain

$$(x_k - t^*)^{\Gamma+1} \leq C [n^{-\Gamma-1} + \frac{1}{n} (x_k - t^*)^\Gamma].$$

Hence  $x_k \leq t^* + C \frac{1}{n}$ . In the first case of (14) we have

$$(x_k - t^*)^{\Gamma+1} - (x_{k+1} - t^*)^{\Gamma+1} \leq C \frac{1}{n} [(x_k - t^*)^\Gamma + (x_{k+1} - t^*)^\Gamma].$$

Thus

$$x_k - x_{k+1} \leq C \frac{1}{n} \frac{[(x_k - t^*) - (x_{k+1} - t^*)][(x_k - t^*)^\Gamma + (x_{k+1} - t^*)^\Gamma]}{(x_k - t^*)^{\Gamma+1} - (x_{k+1} - t^*)^{\Gamma+1}}$$

and

$$x_k - x_{k+1} \leq C \frac{1}{n} \sup_{1 < x < \infty} \frac{(x-1)(x^\Gamma+1)}{x^{\Gamma+1}-1} \leq C \frac{1}{n}$$

since  $\Gamma+1 > 0$ . The other possibilities may be treated similarly. To estimate

$x_k - x_{k+1}$  from below let us remark as Erdős-Turán did that

$$1 = (x_{kn} - x_{k+1,n}) \frac{d}{dx} t_{kn}^2(d\alpha, x)$$

where  $x_{k+1} \leq x^* \leq x_k$ . We have

$$t_{kn}^2(d\alpha, x) \leq \lambda_{kn}(d\alpha) \lambda_{n-1}^{-1}(d\alpha, x).$$



Using Theorems 19 and 6.3.25 we obtain

$$\left| \frac{d}{dx} \ell_{kn}^2(d\alpha, x) \right| \leq Cn$$

uniformly for  $x_{kn}, x \in \Delta_1 \subset \Delta^0$ . Hence the estimate from below for  $x_k - x_{k+1}$  follows.

**Theorem 21.** Let  $\text{supp}(d\alpha)$  be compact,  $\Delta(d\alpha) = [c_1, c_2]$ ,  $a > -1$ ,  $\delta > 0$ .

Let  $\alpha$  be absolutely continuous in  $[c_2 - \delta, c_2]$  and let

$$\alpha'(t) \sim (c_2 - t)^a$$

for  $t \in [c_2 - \delta, c_2]$ . Let  $x_{kn}(d\alpha) = \frac{1}{2}(c_1 + c_2) + \frac{1}{2}(c_2 - c_1) \cos \theta_k$  for  $k = 0, 1, \dots, n+1$  where  $0 \leq \theta_k \leq \pi$  and  $x_{0n} = c_2$ ,  $x_{n+1, n} = c_1$ . Then

$$\theta_{k+1} - \theta_k \sim \frac{1}{n}$$

for  $x_{kn} \in \Delta^-(c_2 - \delta, c_2)$ .

**Proof.** We can assume without loss of generality that  $\Delta(d\alpha) = [-1, 1]$  and  $\delta \leq \frac{1}{2}$ . (Concerning the second assumption see e.g. Freud, §III. 5.) We obtain immediately from Theorem 6.3.27 and Markov-Stieltjes' inequalities that

$\theta_1 = O(\frac{1}{n})$  and  $\theta_2^{-1} = O(n)$ . Now we shall show that  $\theta_1^{-1} = O(n)$ . Let  $m \geq n$  be fixed. Then by the Gauss-Jacobi mechanical quadrature formula

$$\begin{aligned} (1-x_{kn}) \lambda_{kn}(d\alpha) &= \int_{-1}^1 (1-t) \ell_{kn}^2(d\alpha, t) d\alpha(t) = \\ &= \sum_{k=1}^m (1-x_{km}) \ell_{km}^2(d\alpha, x_{km}) \lambda_{km}(d\alpha). \end{aligned}$$

Hence

$$\begin{aligned} (1-x_{kn}) \lambda_{kn}(d\alpha) &\geq (1-x_{2m}) \sum_{k=1}^m \ell_{kn}^2(d\alpha, x_{km}) \lambda_{km}(d\alpha) - \\ &\quad - (1-x_{2m}) \ell_{kn}^2(d\alpha, x_{1m}) \lambda_{1m}(d\alpha) = \\ &= (1-x_{2m}) \lambda_{1n}(d\alpha) \left( 1 - \frac{\ell_{kn}^2(d\alpha, x_{1m}) \lambda_{1m}(d\alpha)}{\lambda_{1n}(d\alpha)} \right) \end{aligned}$$

and consequently

$$1 - x_{1n} \geq (1-x_{2m}) \left[ 1 - \frac{\lambda_{1m}(d\alpha)}{\lambda_{1n}(d\alpha, x_{1m})} \right].$$

Putting here  $m = Nn$  where  $N$  is big but fixed we obtain from Theorem 6.3.27 that

$$1 - x_{1n} \geq \frac{1}{2}(1-x_{2m})$$

if only  $n$  is big enough. Hence  $\theta_1^{-1} = O(n)$ . To prove  $\theta_{k+1} - \theta_k = O(n^{-1})$  for  $x_k \in \Delta \subset (1-\delta, 1]$  we shall use the inequalities

$$\begin{aligned} (17) \quad \sum_{k=1+1}^n (1+x_{kn}) \lambda_{kn}(d\alpha) &\leq \int_{-1}^{x_{1n}} (1+t) d\alpha(t) \leq \\ &\leq \sum_{k=1}^n (1+x_{kn}) \lambda_{kn}(d\alpha) \quad (1=1, 2, \dots, n) \end{aligned}$$

which is always true if  $\Delta(d\alpha) \subset [-1, 1]$ . We shall not prove (17), it can be proved in the same way that Freud proves the Markov-Stieltjes inequalities in his book. From (17) and Theorem 6.3.27 we get for  $x_k \in \Delta$

$$\theta_{k+1} - \theta_k \leq \frac{C}{n} \sup_{0 \leq x, y \leq \frac{\pi}{2}} \frac{|x-y| \left| \sin \frac{2a+3}{2a+4} x + \sin \frac{2a+3}{2a+4} y \right|}{\left| \sin \frac{2a+4}{2a+4} x - \sin \frac{2a+4}{2a+4} y \right|}$$

which is of order  $1/n$  since  $2a+3 > 1$ . The estimate  $[\theta_{k+1} - \theta_k]^{-1} = O(n)$  for  $x_k \in \Delta$  follows from Lemma 14 and Theorem 6.3.27 in the same way as we obtained the estimates from below in Theorem 20.

**Theorem 22.** Let  $w$  be as in Theorem 16,  $x_{kn}(w) = \cos \theta_{kn}$  ( $x_{0n} = 1$ ,  $x_{n+1, n} = -1$ ,  $0 \leq \theta_{kn} \leq \pi$ ). Then

$$\theta_{k+1, n} - \theta_{kn} \sim \frac{1}{n}$$

for  $k = 0, 1, \dots, n$ .

**Proof.** Use Theorems 20 and 21.

The following inequalities will play a fundamental role in investigations of mean convergence of interpolation processes.

**Theorem 23.** Let  $\alpha, \Delta, t, \Gamma$  and  $\Delta_1$  be as in Theorem 20. Let  $1 \leq p < \infty$  and  $m \leq c_1 n$ . Then for each  $\pi_m$  and  $n$

$$\sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq C \int_{\Delta} |\pi_m(t)|^p d\alpha(t)$$

where  $C = C(\alpha, \Delta_1, p, c_1)$ .

**Proof.** Let  $w = 1/\alpha'$  with  $\text{supp}(w) = \Delta$ . Then by Theorem 6.3.25

$$\lambda_{kn}(d\alpha) \sim \lambda_n(w, x_{kn}(d\alpha)) \sim \lambda_n(w, p, x_{kn}(d\alpha)) \sim \lambda_{n+m}(w, p, x_{kn}(d\alpha))$$

for  $x_{kn}(d\alpha) \in \Delta_1 \subset \Delta^0$  since  $m \leq c_1 n$ . Hence

$$|\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq C \int_{\Delta} |\pi_m(t)|^p d\alpha(t)$$

for  $x_{kn}(d\alpha) \in \Delta_1$ . Further we can suppose that  $t \in \Delta_1$ . Let  $j = j(n)$  be defined by  $x_{j+1, n} \leq t < x_{j, n}$ . Consider

$$\sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) = \sum_{k < j-1}$$

Observe that

$$|\pi_m(x_{kn})|^p \leq |\pi_m(x)|^p + p \int_{x_{k+1, n}}^{x_{k-1, n}} |\pi_m(t)|^{p-1} |\pi'_m(t)| dt$$

for  $x_{k+1, n} \leq x \leq x_{k-1, n}$ . Thus by the Markov-Stieltjes inequalities we obtain

$$\begin{aligned} \sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) &\leq 2 \int_{\Delta} |\pi_m(x)|^p d\alpha(x) + \\ &+ p \sum_{x_{kn} \in \Delta_1} \lambda_{kn}(d\alpha) \int_{x_{k+1, n}}^{x_{k-1, n}} |\pi_m(t)|^{p-1} |\pi'_m(t)| dt. \end{aligned}$$

-211-

By Theorems 20 and 6.3.25

$$\lambda_{kn}(d\alpha) \sim \frac{1}{n} |t-t^*| \Gamma$$

for  $x_{k+1, n} \leq t \leq x_{k-1, n}$  if  $k < j-1$  and  $x_{kn} \in \Delta_1$ . Consequently

$$\begin{aligned} \sum_{x_{kn} \in \Delta_1} \lambda_{kn}(d\alpha) \int_{x_{k+1, n}}^{x_{k-1, n}} |\pi_m(t)|^{p-1} |\pi'_m(t)| dt &\leq \\ &\leq C \frac{1}{n} \left\{ \int_{\Delta_1(\varepsilon)} |\pi_m(t)|^p |t-t^*| \Gamma dt \right\}^{\frac{p-1}{p}} \cdot \left\{ \int_{\Delta_1(\varepsilon)} |\pi'_m(t)|^p |t-t^*| \Gamma dt \right\}^{\frac{1}{p}}, \end{aligned}$$

where  $\varepsilon > 0$  is chosen so that  $\Delta_1(\varepsilon) \subset \Delta^0$ . By Lemma 15 we obtain

$$\sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq C \frac{m+n}{n} \int_{\Delta} |\pi_m(t)|^p d\alpha(t).$$

The sum for  $x_{kn} \in \Delta_1$ ,  $k > j-1$  can be estimated similarly.

**Theorem 24.** Let  $\alpha, c_2, a, \delta$  and  $\Delta$  be as in Theorem 21. Let  $\Gamma > -1-a$ ,  $1 \leq p < \infty$ ,  $m \leq c^* n$ . Then

$$\begin{aligned} \sum_{x_{kn} \in \Delta} |\pi_m(x_{kn})|^{p(c_2-x_{kn})} \Gamma \lambda_{kn}(d\alpha) &\leq \\ &\leq C \int_{c_2-\delta}^{c_2} |\pi_m(t)|^{p(c_2-t)} \Gamma d\alpha(t) \end{aligned}$$

for each  $\pi_m$  where  $C = C(\alpha, p, \Delta, \Gamma, c^*)$ .

**Theorem 25.** Let  $w$  be as in Theorem 16,  $u \in L_w^1$  be a Jacobi weight,  $1 \leq p < \infty$  and  $m \leq c^* n$ . Then

$$\sum_{k=1}^n |\pi_m(x_{kn})|^{p(u(x_{kn}) \lambda_{kn}(w))} \leq C \int_{-1}^1 |\pi_m(t)|^p u(t) w(t) dt$$

for every  $\pi_m$  where  $C = C(w, u, p)$ .

Theorems 24 and 25 can be proved by the same method as Theorem 23.

As an application of the previous results we shall prove two theorems.

-212-

Theorem 26. Let  $w = w(a, b)$  be the Pollaczek weight defined in Definition 6.2.12. Let  $1 \leq p \leq \infty$ ,  $p \neq 2$ . Then the sequence of operators  $\{S_n(w)\}$  is not uniformly bounded in  $L_w^p$ .

Proof. By Corollary 13 and Theorem 23 ( $\Gamma = 0$ )

$$\liminf_{n \rightarrow \infty} \int_{-1}^1 |p_n(w, t)|^q w(t) dt > 0$$

for  $1 \leq q < \infty$ . Let  $1 < p < \infty$ ,  $p \neq 2$ . Suppose that

$$\sup_{n \geq 1} \|S_n(w)\|_{L_w^p \rightarrow L_w^p} < \infty.$$

Then we obtain exactly in the same way as in Theorem 8.13 that

$$\sup_{n \geq 1} \|p_n(w)\|_{w, p} < \infty$$

and

$$\sup_{n \geq 1} \|p_n(w)\|_{w, p'} < \infty$$

where  $p' = \frac{p}{p-1}$ . Since either  $p$  or  $p'$  is greater than 2 this cannot happen by Theorem 7.31. For  $p = 1$  or  $p = \infty$  the Theorem follows from old results. (See e.g. Freud, remarks on Chapter IV.)

Theorem 27. Theorem 26 remains valid if we replace the Pollaczek weight there by the weight defined in Example 6.2.14.

Proof. By Korovus' theorem (See Freud, §1.7.) the corresponding system is uniformly bounded on each  $\Delta \subset (-1, 1)$ . Now we can repeat the proof of Theorem 26.

Definition 28. The weight  $w$  is a generalized Jacobi weight ( $w \in GJ$ ) if

$$\text{supp}(w) = [-1, 1] \text{ and } w(t) = \varphi(t) \prod_{k=2}^{N-1} |t - t_k|^{-\Gamma_k} \prod_{k=2}^N |t - t_k|^{-\Gamma_k}$$

where  $\Gamma_k > -1$  ( $k = 1, 2, \dots, N$ ),  $1 > t_2 > \dots > t_{N-1} > -1$ ,  $\varphi > 0$  is continuous

on  $[-1, 1]$  and  $\omega(\delta)/\delta \in L^1(0, 1)$  where  $\omega$  is the modulus of continuity of  $\varphi$ .

If  $w^* \sim w$  where  $w \in GJ$  then we write  $w^* \sim GJ$ . Hence if  $w \in GJ$

then also  $w \sim GJ$ . For  $w \sim GJ$   $w_n$  is defined by

$$w_n(t) = (\sqrt{1-t} + \frac{1}{n}) \prod_{k=2}^{N-1} (|t_k - t| + \frac{1}{n})^{-\Gamma_k} (\sqrt{1+t} + \frac{1}{n})^{-2\Gamma_N}$$

( $-1 \leq t \leq 1$ ;  $n = 1, 2, \dots$ ).

Lemma 29. Let  $w \in GJ$ . Then

$$(\sqrt{1-x} + \frac{1}{n})(\sqrt{1+x} + \frac{1}{n}) w_n(x) p_n^2(w, x) \leq C$$

for  $-1 \leq x \leq 1$ ,  $n = 1, 2, \dots$  where  $C \neq C(n, x)$ .

Proof. See Badkov [2].

Lemma 30. Let  $\alpha \in S$ ,  $g = v^{-2}$ . Then

$$|p_{n-1}(d\alpha, x_k)| \sim (1-x_k^2)^{1/2} |p_{n-1}(d\alpha_g, x_k)|$$

where  $x_k = x_{kn}(d\alpha)$ .

Proof. In course of the proof of Theorem 4.2.3 we have shown that

$$(1-x_k^2) p_{n-1}^2(d\alpha_g, x_k) = p_{n-1}^2(d\alpha, x_k) \cdot \left[ \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\alpha_g)} + \frac{\gamma_{n-1}(d\alpha) \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)} \right]$$

The expression in the brackets does not depend on  $k$  and by Lemma 4.2.2 it converges to 1 when  $n \rightarrow \infty$ .

Theorem 31. Let  $w \in GJ$ . Then

$$w(x_{kn}) p_{n-1}^2(w, x_{kn}) \sim \sqrt{1-x_{kn}^2} \sim (\sqrt{1-x_{kn}^2} + \frac{1}{n}) (\sqrt{1-x_{kn}^2} + \frac{1}{n}).$$

Proof. If  $w \in GJ$  then  $v^{-2} w \in GJ$ . Hence by Lemmas 29 and 30

$$w(x_{kn}) p_{n-1}^2(w, x_{kn}) \leq C (\sqrt{1-x_{kn}^2} + \frac{1}{n}) (\sqrt{1-x_{kn}^2} + \frac{1}{n}).$$

By Theorem 22 the right side here is  $\sim \sqrt{1-x_{kn}^2}$ . The converse inequality



follows from

$$p_{n-1}(w, x_{kn})^{-1} = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \lambda_{kn}(w) p_n(w, x_{kn})$$

and from Theorems 6.3.28, 19, 22 and Lemma 29.

Lemma 32. Let  $\alpha$  be an arbitrary weight. Let  $x_{0n}(d\alpha) = \infty$ ,  $x_{n+1,n}(d\alpha) = -\infty$ ,  $f_{0n}(d\alpha, x) \equiv 0$  and  $f_{n+1,n}(d\alpha, x) \equiv 0$ . Let  $x_{k+1,n}(d\alpha) \leq x \leq x_{kn}(d\alpha)$  ( $k = 0, 1, \dots, n$ ). Then

$$f_{kn}(d\alpha, x) + f_{k+1,n}(d\alpha, x) \geq 1.$$

Proof. See Erdős-Turán [5].

Theorem 33. Let  $w \in GJ$ . Then

$$(18) \quad \lambda_n(w, x) p_n^2(w, x) \sim n(x - x_k)^2 (\sqrt{1-x} + \frac{1}{n})^{-2} (\sqrt{1-x} + \frac{1}{n})^{-2}$$

for  $-1 \leq x \leq 1$  where  $x_k$  is the zero of  $p_n(w, x)$  which is closest to  $x$ .

Proof. By Theorems 22 and 6.3.28

$$f_{kn}^2(w, x) \leq \lambda_{kn}(w) \lambda_n^{-1}(w, x) \leq C$$

for  $-1 \leq x \leq 1$  where  $k$  is the index of  $x_k$  in (18). Further, if  $k = 1$  and

$x_k \leq x \leq 1$  or  $k = n$  and  $-1 \leq x \leq x_k$  then by Lemma 32  $f_{kn}^2(w, x) \geq 1$ . Other-

wise  $x$  is between either  $x_{k-1}$  and  $x_k$  ( $k > 1$ ) or  $x_k$  and  $x_{k+1}$  ( $k < n$ ).

Let for simplicity  $x_k \leq x \leq x_{k+1}$ . Then by Lemma 32

$$f_{k-1,n}(w, x) + f_{k+1,n}(w, x) \geq 1.$$

By Theorems 22, 31 and 6.3.28

$$|\lambda_{k-1,n}(w) p_{n-1}(w, x_{k-1,n})| \sim |\lambda_{kn}(w) p_{n-1}(w, x_{kn})|$$

and obviously  $f_{k-1,n}(w, x) \geq 0$ ,  $f_{kn}(w, x) > 0$  and sign  $p_{n-1}(w, x_{k-1,n}) =$   
 $= -\text{sign } p_{n-1}(w, x_{kn})$ . Hence

$$f_{k-1,n}(w, x) \leq C f_{kn}(w, x).$$

Consequently in all possible cases

$$f_{kn}^2(w, x) \sim 1.$$

The Theorem follows now from Theorems 22, 31 and 6.3.28.

Corollary 34. Let  $w \in GJ$ . Then

$$p_n(w, 1) \sim n^{\frac{1}{2}}$$

and

$$|p_n(w, -1)| \sim n^{\frac{1}{2}}$$

Proof. In this case either  $k = 1$  or  $k = n$ .

Remark 35. Let  $\alpha, a$  and  $c_2$  be as in Theorem 21. Then  $p_n(d\alpha, c_2) \geq C n^a$

for  $n = 1, 2, \dots$  where  $C \neq C(n)$ . This follows immediately from Theorems

21, 6.3.27 and from  $\lambda_n^{-1}(d\alpha, x) = \sum_{k=1}^n \lambda_{kn}^{-1}(d\alpha) f_{kn}^2(d\alpha, x)$ .

Lemma 36. Let  $\text{supp}(d\alpha)$  be compact. Let there exist an interval  $\tau$  such

that the sequence  $\{|p_n(d\alpha, x)|\}$  is uniformly bounded for  $x \in \tau$ . Then

$$\liminf_{n \rightarrow \infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} > 0.$$

Proof. By Theorem 7.5  $\tau \subset \text{supp}(\alpha') \subset \text{supp}(d\alpha)$ . Let  $\tau = [c_1, c_2]$  and

$m_1 = m_1(n)$ ,  $m_2 = m_2(n)$  be defined by  $x_{m_1+1,n} < c_1 \leq x_{m_1,n}$  and

$x_{m_2,n} \leq c_2 < x_{m_2-1,n}$  respectively where  $x_{n+1,n} = -\infty$ ,  $x_{0n} = \infty$ . By Lemma

3.2.2  $\lim_{n \rightarrow \infty} x_{m_1,n} = c_1$  and  $\lim_{n \rightarrow \infty} x_{m_2,n} = c_2$ . We can suppose that  $\alpha$  is

continuous at  $c_1$  and  $c_2$ . If not we can replace  $\tau$  by a smaller interval.

Let  $m_2 < k \leq m_1$ . Then by Lemma 32

$$f_{k-1,n}(d\alpha, \frac{x_{k-1} + x_k}{2}) + f_{kn}(d\alpha, \frac{x_{k-1} + x_k}{2}) \geq 1,$$

that is



$$x_{k-1} - x_k \leq \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} C^2 \lambda_{k-1,n}(d\alpha) + \lambda_{kn}(d\alpha)$$

where  $C = \sup_{n \geq 0} \max_{x \in \tau} |p_n(d\alpha, x)|$ . Hence

$$x_{m_2} - x_{m_1} \leq C^2 \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \sum_{k=m_2}^{m_1} \lambda_{kn}(d\alpha).$$

Letting  $m_1 \rightarrow \infty$  we obtain

$$|\tau| \leq 4C^2 \liminf_{n \rightarrow \infty} \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \int_{\tau} d\alpha(t).$$

**Theorem 37.** Let  $\text{supp}(d\alpha) \subset [-1, 1]$ ,  $\text{supp}(d\beta) \subset [-1, 1]$ ,  $\tau \subset (-1, 1)$  and  $\sigma\tau = [-1, 1] \setminus \tau$ . Let  $d\alpha(t) = d\beta(t)$  for  $t \in \tau$ ,  $\alpha$  be absolutely continuous in  $\sigma\tau$  and let there exist two polynomials  $\pi_1$  and  $\pi_2$  such that  $\pi_1 \alpha' / \beta' \in L_{d\alpha}^1(\sigma\tau)$  and  $|\pi_2 \alpha' p_n(d\alpha, x)| \leq K$  for  $x \in \sigma\tau$  and  $n=1, 2, \dots$ . Then

$$|t_n(d\beta, x)| \leq C[|p_n(d\alpha, x)| + |p_{n-1}(d\alpha, x)|]$$

uniformly for  $x \in \tau_1 \subset \tau^0$  and  $n=1, 2, \dots$ .

**Proof.** We can suppose that neither  $\pi_1$  nor  $\pi_2$  has zeros in  $\tau^0$ . Let

$$\rho = \pi_1^{-1} \pi_2, \deg \rho = m. \text{ Then}$$

$$p(x)p_n(d\beta, x) = \int_{-1}^1 p(t)p_n(d\beta, t)K_{n+m+1}(d\alpha, x, t)d\alpha(t).$$

By the conditions  $\int_{-1}^1 ( ) d\alpha = \int_{-1}^1 ( ) d\beta + \int_{\sigma\tau} ( ) d\alpha - \int_{\sigma\tau} ( ) d\beta$ . Let  $x \in \tau_1 \subset \tau^0$ .

Then

$$\begin{aligned} \left\{ \int_{\sigma\tau} p(t)p_n(d\beta, t)K_{n+m+1}(d\alpha, x, t)d\beta(t) \right\}^2 &\leq \\ &\leq C[|p_{n+m}(d\alpha, x)| + |p_{n+m+1}(d\alpha, x)|]^2 \int_{-1}^1 \pi_1^2(t)d\beta(t) \end{aligned}$$

and

$$\begin{aligned} \left\{ \int_{\sigma\tau} p(t)p_n(d\beta, t)K_{n+m+1}(d\alpha, x, t)d\alpha(t) \right\}^2 &\leq \\ &\leq C[|p_{n+m}(d\alpha, x)| + |p_{n+m+1}(d\alpha, x)|]^2 \cdot \\ &\cdot \int_{\sigma\tau} p^2(t) \frac{\alpha'(t)}{\beta'(t)} dt \int_{\sigma\tau} p_n^2(d\beta, t) \beta'(t) dt. \end{aligned}$$

Further for  $n > m$

$$\begin{aligned} \int_{-1}^1 p(t)p_n(d\beta, t)K_{n+m+1}(d\alpha, x, t)d\beta(t) &= \\ \int_{-1}^1 p(t)p_n(d\beta, t) \sum_{k=n-m}^{n+m} [p_k(d\alpha, x)p_k(d\alpha, t)]d\beta(t). \end{aligned}$$

Consequently

$$\begin{aligned} \left| \int_{-1}^1 p(t)p_n(d\beta, t)K_{n+m+1}(d\alpha, x, t)d\beta(t) \right| &\leq \\ &\leq C \sum_{k=n-m}^{n+m} |p_k(d\alpha, x)| \left\{ \int_{-1}^1 p_k^2(d\alpha, t)d\beta(t) \right\}^{\frac{1}{2}}. \end{aligned}$$

But

$$\begin{aligned} \int_{-1}^1 p(t)^2 p_k^2(d\beta, t)d\beta(t) &= \int_{\tau} p(t)^2 p_k^2(d\alpha, t)d\alpha(t) + \\ &+ \int_{\sigma\tau} p(t)^2 p_k^2(d\alpha, t)d\beta(t) \leq C. \end{aligned}$$

Thus we have proved that for  $x \in \tau_1 \subset \tau^0$

$$|p_n(d\beta, x)| \leq C \sum_{k=n-m}^{n+m+1} |p_k(d\alpha, x)|.$$

By Lemma 36 the sequence  $\left\{ \frac{Y_{n-1}(d\alpha)}{Y_n(d\alpha)} \right\}$  is bounded from below by a positive constant. Thus by Lemma 3.3.1 and by the recurrence formula

$$|p_k(d\alpha, x)| \leq C[|p_n(d\alpha, x)| + |p_{n-1}(d\alpha, x)|]$$

whenever  $n \geq 1$ ,  $n-k = O(1)$  and  $x \in \Delta$  where  $C = C^*([n-k], \alpha, \Delta)$ .

**Remark 38.** Theorem 37 becomes useful if we combine it with Korovskii's theorem and with results in Freud and Geronimus.

Theorem 39. Theorem 37 remains valid if  $\tau$  is of the form  $[a, 1]$  or  $[-1, a]$  where  $|a| < 1$  and  $\tau_1 \subset (a, 1]$  or  $\tau_1 \subset [-1, a)$  respectively.

Proof. The same as that of Theorem 37.

# 10. Lagrange Interpolation.

First we shall consider the Lebesgue function  $L_n(da, x)$  of Lagrange interpolation corresponding to  $\alpha$  which is defined by

$$L_n(da, x) = \sum_{k=1}^n |l_{kn}(da, x)|.$$

The estimate

$$(1) \quad L_n^2(da, x) \leq \lambda_n^{-1}(da, x) [\alpha(\infty) - \alpha(-\infty)]$$

shows that if  $\alpha$  has a jump at  $x$  then

$$L_n^2(da, x) \leq \frac{\alpha(\infty) - \alpha(-\infty)}{\alpha(x+0) - \alpha(x-0)}$$

but in general (1) is not a very strong result. Our aim will be to improve (1).

Lemma 1. Let  $\text{supp}(da)$  be compact and  $\epsilon > 0$ . Then

$$\lambda_n(da, x) L_n^2(da, x) \leq 2 \sum_{|x - x_{kn}| < \epsilon} \lambda_{kn}(da) + C \epsilon^{-2} \lambda_n(da, x) p_n^2(da, x)$$

for  $x \in \Delta$ ,  $n=1, 2, \dots$  where  $C = C(\alpha, \Delta)$ .

Proof. Repeat the proof of Lemma 8.6 and apply Lemma 3.3.1.

Theorem 2. Let  $\alpha \in M(0, 1)$ . If  $\alpha$  is continuous at  $x \in [-1, 1]$  then

$$(2) \quad \lim_{n \rightarrow \infty} \lambda_n(da, x) L_n^2(da, x) = 0.$$

If  $\alpha$  is continuous on the closed set  $\mathfrak{D} \subset (-1, 1)$  then (2) holds uniformly for  $x \in \mathfrak{D}$ .

Proof. Apply Theorem 4.1.11, Lemmas 9.1, 9.7 and 1.

Theorem 3. Let  $\alpha \in M(0, 1)$ ,  $\tau \subset [-1, 1]$ . If  $[\alpha']^{-\epsilon} \in L^1(\tau)$  with some  $\epsilon > 0$  then

$$(3) \quad \lim_{n \rightarrow \infty} n^{-\frac{1}{3}} L_n(da, x) = 0$$

for almost every  $x \in \tau$ . If  $\alpha'(t) \geq c > 0$  for almost every  $t \in \tau$  and  $\alpha$  is continuous on  $\tau$  then (3) is satisfied uniformly for  $x \in \tau_1 \subset \tau$ .

Proof. Repeat the reasoning in the proof of Theorem 8.9.

Corollary 4. Convergence theorems for  $Lip_1^*$ .

In the following we shall investigate  $L_n^*(da, x)$  defined by

$$L_n^*(da, x)^2 = \frac{1}{n} \sum_{k=1}^n L_k^2(da, x).$$

If we can estimate  $L_n^*(da, x)$  we can also estimate the Lebesgue function of the  $(C, 1)$  means of Lagrange interpolating polynomials, which we denote by  $\hat{L}_n(da, x)$ , since obviously  $\hat{L}_n(da, x) \leq L_n^*(da, x)$ . Moreover, we can also estimate the convergence rate of the strong  $(C, 1)$  means:

$$\sigma_n(da, f, x) = \frac{1}{n} \sum_{k=1}^n |f(x) - L_k(da, f, x)|.$$

Let  $E_k(f)$  denote the best approximation of  $f$  by  $\pi_{k-1}$  in  $C(d\alpha)$ . Then

$$\sigma_n(da, f, x) \leq \frac{1}{n} \sum_{k=1}^n [1 + L_k(da, x)] E_k(f).$$

Hence by Jackson's theorem

$$\sigma_n(f, da, x) \leq C \cdot L_n^*(da, x) \left\{ \frac{1}{n} \int_1^1 \frac{\omega_R(f, t)^2}{t^2} dt \right\}^{\frac{1}{2}}$$

where  $\omega_R$  denotes the  $R$ -th modulus of smoothness of  $f$ .

Theorem 5. Let  $\text{supp}(d\alpha)$  be compact,  $\tau \subset \text{supp}(d\alpha)$ ,  $\varepsilon > 0$  and  $[\alpha']^{-\varepsilon} \in L^1(\tau)$ . Then

$$(4) \quad \limsup_{n \rightarrow \infty} n^{-\frac{1}{3}} L_n^*(da, x) < \infty$$

for almost every  $x \in \tau$ . If  $\alpha \in Lip_X$  and  $\alpha'(t) \geq c > 0$  for  $|x-t|$  small then

(4) holds. If  $\alpha \in Lip_1$  on  $\tau$  and  $\alpha'(t) \geq c > 0$  for  $t \in \tau$  then (4) is satisfied uniformly for  $x \in \tau_1 \subset \tau$ .

Proof. For simplicity let us prove the first part of the Theorem. Let  $x \in \tau_1 \subset$

$\subset \tau^0$  and  $\varepsilon = n^{-\frac{1}{3}}$ . Then by Theorem 9.6 and Lemma 1

$$\begin{aligned} \lambda_n(da, x) L_n^2(da, x) &\leq 2 \left[ \alpha(x+cn^{-\frac{1}{3}}) - \alpha(x-cn^{-\frac{1}{3}}) \right] + \\ &+ C_1 n^{-\frac{2}{3}} \lambda_n(da, x) p_n^2(da, x). \end{aligned}$$

Hence

$$\begin{aligned} L_n^*(da, x)^2 &\leq \\ &\leq 2 n^{-\frac{1}{3}} \sum_{k=1}^n \frac{1}{k \lambda_k(da, x)} \left[ \frac{1}{k^{\frac{1}{3}}} \left( \alpha(x+ck^{-\frac{1}{3}}) - \alpha(x-ck^{-\frac{1}{3}}) \right) \right] \\ &+ C_1 n^{-\frac{1}{3}} \lambda_{n+1}^{-1}(da, x). \end{aligned}$$

Since  $\alpha$  is almost everywhere differentiable we obtain from Theorem 6.3.25 that (3) holds for almost every  $x \in \tau_1$ . But  $\tau_1 \subset \tau^0$  is arbitrary.

Let us note that the second part of Theorem 5 is not new. (See Freud, Some unsolved problems.)

Corollary 6. Convergence of  $(C, 1)$  and strong  $(C, 1)$  means of Lagrange interpolation polynomials for  $f \in Lip_{\frac{1}{3}}$ .

Corollary 7. If  $\text{supp}(d\alpha)$  is compact and  $[\alpha']^{-\varepsilon} \in L^1(\tau)$  with some  $\varepsilon > 0$  then

$$\liminf_{n \rightarrow \infty} n^{-\frac{1}{3}} L_n(da, x) < \infty$$

for almost every  $x \in \tau$ .

Theorem 8. Let  $\alpha \in S$  and  $x \in [-1, 1]$ . Then

$$|p_n(da, x)| \leq 4 \sqrt{\frac{2}{\pi}} T_n(\alpha, 0)^{-1} L_n(da, x)$$

for  $n=1, 2, \dots$ .



Proof. We obtain from Lemma 6.1.19 and from the inequality between the arithmetic and geometric means that

$$Y_{n-1}(d\alpha) \leq 2^{n-1} \sqrt{\frac{2}{\pi}} D(d\alpha, 0)^{-1}.$$

Further

$$\begin{aligned} \frac{2^{n-2}}{Y_{n-1}(d\alpha)} &= \int_{-1}^1 T_{n-1}(t) p_{n-1}(d\alpha, t) d\alpha(t) \leq \\ &\leq 2 \sum_{k=1}^n \lambda_{kn}(d\alpha) \frac{|p_{n-1}(d\alpha, x_{kn})|}{|x - x_{kn}|} \end{aligned}$$

for  $-1 \leq x \leq 1$ . Hence the Theorem follows.

Remark 9. In general Theorem 8 cannot be improved. Let  $w \sim GJ$  with

$\Gamma_1 > 0$ . Then  $w \in S$  and

$$\begin{aligned} \left\{ \sum_{k=1}^n \lambda_{kn}(w) \frac{|p_{n-1}(w, x_{kn})|}{1 - x_{kn}} \right\}^2 &\leq \\ &\leq 2 \sum_{k=1}^n \lambda_{kn}(w) \frac{p_{n-1}^2(w, x_{kn})}{1 - x_{kn}^2} \sum_{k=1}^n \lambda_{kn}(w) (1 - x_{kn})^{-1} \end{aligned}$$

which is bounded by Theorems 4.2.3, 6.3.28 and 9.22.

Definition 10. Let  $x_{0n}(d\alpha) = \infty$ ,  $x_{n+1,n}(d\alpha) = -\infty$ ,  $f_{0n}(d\alpha, x) = f_{n+1,n}(d\alpha, x) = 0$ . Let  $m = m(n, x)$  be defined by  $x_{m+1,n}(d\alpha) < x \leq x_{mn}(d\alpha)$ . Then we put

$$\tilde{L}_n(d\alpha, x) = \sum_{k=1}^n \sum_{m+1}^{\infty} |f_{kn}(d\alpha, x)|.$$

Lemma 11. Let  $\alpha$  be an arbitrary weight. Then

$$L_n(d\alpha, x) - 1 \sim \tilde{L}_n(d\alpha, x)$$

for  $x \in \mathbb{R}$  and  $n=1, 2, \dots$

Proof. By Lemma 9.32

$$\tilde{L}_n(d\alpha, x) + 1 \leq L_n(d\alpha, x).$$

On the other hand, since  $f_{mn}(d\alpha, x) \geq 0$  and  $f_{m+1,n}(d\alpha, x) \geq 0$ , we have

$$L_n(d\alpha, x) = \tilde{L}_n(d\alpha, x) + 1 - \sum_{\substack{k=1 \\ k \neq m, m+1}}^n f_{kn}(d\alpha, x) \leq 1 + 2\tilde{L}_n(d\alpha, x).$$

Recall that  $GJ$  and  $w_n$  have been defined in Definition 9.28.

Theorem 12. Let  $w \in GJ$ . Then

$$(5) \quad L_n(w, x) - 1 \sim \sim \frac{n|x-x_j|}{\sqrt{1-x^2} + \frac{1}{n}} \int \frac{n|t(\sqrt{1-t^2} + \frac{1}{n})^{\frac{1}{2}}}{w_n(x)(\sqrt{1-x^2} + \frac{1}{n})^{\frac{1}{2}} |x-t|} dt$$

for  $-1 \leq x \leq 1$  where  $x_j$  denotes the zero of  $p_n(w)$  closest to  $x$ .

Proof. The Theorem follows by calculation from Theorems 6.3.28, 9.22, 9.31, 9.33 and Lemma 11.

For the case when  $w$  is a Jacobi weight Theorem 12 has been proved by Natanson [10]. The integral on the right hand side of (5) is a rather standard one, it can easily be estimated but the final formula is so complicated that we shall omit it. We shall formulate only one particular case as

Corollary 13. Let  $w \in GJ$ . Then

$$L_n(w, 1) \sim \begin{cases} 1 & \text{for } -1 < \Gamma_1 < -\frac{1}{2} \\ \log n & \text{for } \Gamma_1 = -\frac{1}{2} \\ \frac{\Gamma_1 + \frac{1}{2}}{n} & \text{for } \Gamma_1 > -\frac{1}{2}. \end{cases}$$

Before finding necessary conditions for mean boundedness of Lagrange

interpolation processes let us make some remarks. If we define  $L_n(d\alpha)$  by

$L_n(d\alpha)f = L_n(d\alpha, f)$  then the norm of  $L_n(d\alpha)$  as a mapping from some  $L^q$

( $0 < q < \infty$ ) is never bounded,  $f$  must always be bounded in  $\Delta(d\alpha)^0$ . To

avoid complication, which we cannot solve at the present time, we shall

assume that  $f$  is bounded on  $\Delta(d\alpha)$  and we shall write  $f \in L^\infty(\Delta(d\alpha))$  where



$\|f\|_\infty = \sup_{t \in \Delta(d\alpha)} |f(t)|$ . An important difference between Fourier sums and

Lagrange interpolation polynomials is that  $L_{n+1}(d\alpha, f) - L_n(d\alpha, f)$  is not proportional to  $p_n(d\alpha)$ . If we write

$$L_n(d\alpha, f, x) = \sum_{k=0}^{n-1} a_k p_k(d\alpha, x)$$

and introduce the notation

$$L_{n,k}(d\alpha, f, x) = \sum_{j=0}^{k-1} a_j p_j(d\alpha, x)$$

for  $1 \leq k \leq n-1$  then

$$L_n(d\alpha, f, x) - L_{n,n-1}(d\alpha, f, x) = a_{n-1} p_{n-1}(d\alpha, x)$$

where obviously

$$a_{n-1} = \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) p_{n-1}(d\alpha, x_{kn}).$$

**Theorem 14.** Let either  $\alpha \in S$  or  $\alpha$  satisfy the conditions of Corollary 9.13.

Let  $\beta$  be an arbitrary weight. Let us consider the following three conditions.

$$(i) \sup_{n \geq 1} \|L_n(d\alpha)\|_{L(\Delta(d\alpha))}^p < \infty,$$

$$(ii) \sup_{n \geq 2} \|L_{n,n-1}(d\alpha)\|_{L(\Delta(d\alpha))}^p < \infty$$

and

$$(iii) \sup_{n \geq 1} \|p_{n-1}(d\alpha)\|_{d\beta}^p < \infty$$

where  $p \in (0, \infty)$  is given. Then each pair of (i)-(iii) implies the third one.

**Proof.** Apply Lemma 9.9 and Corollary 9.13.

The following Theorem is one of our main results.

**Theorem 15.** Let  $\alpha \in S$ ,  $0 < p < \infty$  and  $w(\geq 0) \in L^1(-1, 1)$ . Then from

$$\liminf_{n \rightarrow \infty} \|L_n(d\alpha)\|_{L(-1, 1)}^\infty < \infty$$

follows

$$(6) \quad \int_{-1}^1 [\alpha'(t) \sqrt{1-t^2}]^{\frac{p}{2}} w(t) dt < \infty.$$

**Proof.** Let  $\delta = \delta(d\alpha) > 0$  be defined by Theorem 9.10. Let  $\tau \in [-1, 1]$  with  $|\tau| = \frac{\delta}{2}$ . Then we can find a system  $\Omega = \{\tau_1, \tau_2\}$  with  $\tau_1 \tau_2 = \psi$ ,  $|\Omega| >$

$> 2-\delta$  and  $\text{dist}(\tau, \Omega) > 0$  such that (9.7) is satisfied. Let  $f$  be a function on  $[-1, 1]$  which satisfies the conditions  $\|f\|_\infty = 1$  and

$$f(x_{kn}) = 1(x_{kn}) \text{sign}[p_{n-1}(d\alpha, x_{kn})(x_{kn} - B)]$$

where  $B$  is the center of  $\tau$ . Of course  $f$  depends on  $n, \tau, \Omega$ , and  $\alpha$ .

We have

$$\|1_\tau L_n(d\alpha, f)\|_{w,p} \leq \|1_\tau(d\alpha)\|_{L^\infty}^p \rightarrow L_w^p.$$

Since  $|x - x_{kn}| \leq 2$  for  $x \in \tau, x_{kn} \in \Omega$  we obtain

$$\begin{aligned} \|1_\tau p_n(d\alpha)\|_{w,p} &\leq \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha) \|p_{n-1}(d\alpha, x_{kn})\| \leq \\ &\leq 2 \frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} \|1_\tau(d\alpha)\|_{L^\infty}^p \rightarrow L_w^p. \end{aligned}$$

Letting  $n \rightarrow \infty$  we receive from Lemma 4.2.2 and Theorem 9.10 that

$$\liminf_{n \rightarrow \infty} \|1_\tau p_n(d\alpha)\|_{w,p} < \infty.$$

Hence by Theorem 7.32

$$\int_{-1}^1 [\alpha'(t) \sqrt{1-t^2}]^{\frac{p}{2}} w(t) dt < \infty.$$

Since this inequality holds for every  $\tau \in [-1, 1]$  with  $|\tau| = \frac{\delta}{2}$  and  $\delta = \delta(d\alpha) > 0$  it also holds if  $\tau \in [-1, 1]$ .

Using the results of sections 7 and 9 we can prove similar theorems

when  $\alpha \notin S$ . We restrict ourselves to the following

**Theorem 16.** Let  $\text{supp}(d\alpha) = [-1, 1]$ ,  $\alpha'(x) > 0$  for almost every  $x \in [-1, 1]$  and let there exist an interval  $\tau \subset [-1, 1]$  such that the sequence  $\{p_n(d\alpha, x)\}$

is uniformly bounded for  $x \in \tau$ . Let  $w(\geq 0) \in L^1(-1, 1)$ . If  $0 < p < \infty$  and

$$(7) \quad \limsup_{n \rightarrow \infty} \|L_n(d\alpha)\|_{L^p(-1, 1)} < \infty$$

then

$$\limsup_{n \rightarrow \infty} \|p_n(d\alpha)\|_{w, p} < \infty.$$

If  $p \geq 2$  and (7) holds then (6) is satisfied.

Proof. By the conditions  $\{\|L_n p_n(d\alpha)\|_{w, p}\}$  is bounded. Let  $c\tau = [-1, 1] \setminus \tau$ .

Let  $f$  be defined by  $\|f\|_\infty = 1$  and

$$f(x_{kn}) = 1(x_{kn} \in \tau) \operatorname{sign} p_{n-1}(d\alpha, x_{kn}).$$

Then

$$\begin{aligned} \|L_{c\tau} p_n(d\alpha)\|_{w, p} &= \sum_{x_{kn} \in \tau} \lambda_{kn}(d\alpha) |p_{n-1}(d\alpha, x_{kn})| \leq \\ &\leq 2 \frac{\lambda_n(d\alpha)}{\lambda_{n-1}(d\alpha)} \|L_n(d\alpha)\|_{L^\infty} \rightarrow L^p. \end{aligned}$$

Now the Theorem follows from Theorem 7.31, Corollary 9.13 and Lemma 9.36.

Definition 17. Let us say that  $\alpha$  just belongs to  $S$  ( $\alpha \in JS$ ) if  $\alpha \in S$  but for every  $\varepsilon > 0$   $[\alpha]^{-\varepsilon} \notin L^1$ . Example:  $\alpha'(x) = \exp\{-(1-x^2)^{-\delta}\}$  ( $0 < \delta < \frac{1}{2}$ ) or  $\alpha'(x) = \exp\{-|x|^{-\delta}\}$  ( $0 < \delta < 1$ ).

Corollary 18. Let either  $\alpha \in JS$  or  $\alpha$  be a Pollaczek weight or  $\alpha$  be defined by

$$\alpha'(t) = \alpha(t) \exp\{-(1-t^2)^{-\frac{1}{2}}\},$$

$\alpha(> 0) \in Lip1$ ,  $\operatorname{supp}(d\alpha) = [-1, 1]$  and  $\alpha$  is absolutely continuous. Then for every  $p > 2$  there exists a function  $f \in C[-1, 1]$  such that

$$\limsup_{n \rightarrow \infty} \int_{-1}^1 |f(t) - L_n(d\alpha, f, t)|^p \alpha'(t) dt > 0.$$

Proof. Theorems 15, 16 and Banach-Steinhaus' theorem.

Let us remark that Corollary 18 gives a more or less complete answer to Turán's problem and solves Askey's conjecture ([1]). Let us recall that Turán asked if there exists a weight  $\alpha$  with  $\operatorname{supp}(d\alpha) = [-1, 1]$  such that the conclusion of Corollary 18 holds and Askey conjectured that the Pollaczek weight solves Turán's problem.

Theorem 19. Let  $\operatorname{supp}(d\alpha) \subset [-1, 1]$ ,  $0 < p < \infty$ ,  $w(\geq 0) \in L^1(-1, 1)$ . Let the sequence  $n_1 < n_2 \dots$  be given. If for every  $f \in C[-1, 1]$

$$\lim_{k \rightarrow \infty} \int_{-1}^1 |L_{n_k}(d\alpha, f, x) - f(x)|^p w(x) dx = 0$$

then

$$(8) \quad \limsup_{k \rightarrow \infty} \|L_{n_k}(d\alpha)\|_{L^\infty}^p < \infty$$

Proof. If  $p \geq 1$  then the Theorem follows from Banach-Steinhaus' theorem.

Now let  $0 < p < 1$ . Let us define the functionals  $\varphi_k : C[-1, 1] \rightarrow \mathbb{R}$  by

$$\varphi_k(f) = \int_{-1}^1 |L_{n_k}(d\alpha, f, x) - f(x)|^p w(x) dx.$$

Then  $\varphi_k(f+g) \leq \varphi_k(f) + \varphi_k(g)$ ,  $\varphi_k(\lambda f) = |\lambda|^p \varphi_k(f)$ ,  $\varphi_k(f) \geq 0$  and  $\lim_{k \rightarrow \infty} \varphi_k(f) = 0$  for every  $f, g \in C$ . Suppose there exists a subsequence  $k_1 < k_2 < \dots$  such that

$$c_j = \sup_{\|f\|_C \leq 1} \varphi_{k_j}(f) \rightarrow \infty$$

Let us put  $j_1 = 1$  and find a function  $f_1 \in C$  such that  $\|f_1\|_C \leq 1$  and  $\varphi_{k_{j_1}}(f_1) \geq \frac{1}{2} c_1$ . Then there exists a number  $j_2 > j_1$  such that for each  $j \geq j_2$   $\varphi_{k_{j_1}}(f_j) \leq 1$ . Now we find a function  $f_2 \in C$  such that  $\varphi_{k_{j_2}}(f_2) \geq \frac{1}{2} c_{j_2}$  and  $\|f_2\|_C \leq 1$ . After we choose  $j_3 > j_2$  so that  $\varphi_{k_{j_3}}(f_j) < 1$  for every  $j \geq j_3$ . Continuing this process we build up two sequences  $\{j_l\}_{l=1}^\infty$  and  $\{f_l\}_{l=1}^\infty$

so that  $f \in C$ ,  $\|f\|_C \leq 1$ ,  $\varphi_k(f) \geq \frac{1}{2} c_{j_1}$  and  $\varphi_k(f_m) \leq 1$  for  $m = 1, 2, \dots$ ,  $f-1$ . Let us choose a subsequence  $j_1 < j_2 < \dots$  such that  $\sum_{v=1}^{\infty} c_{j_v}^{-p} \leq 1$  and  $c_{j_m} \left\{ \sum_{v=m+1}^{\infty} c_{j_v}^{-1} \right\}^p \leq 1$  for  $m=1, 2, \dots$ . Let  $f = \sum_{v=1}^{\infty} c_{j_v}^{-1} f_{j_v}$ . Then  $f \in C$ . Further for  $m \geq 1$

$$\begin{aligned} \varphi_k(f) &\geq c_{j_m}^{-p} \varphi_k(f_m) - \\ &- \sum_{v=1}^{m-1} c_{j_v}^{-p} \varphi_k(f_{j_v}) - c_{j_m} \left\{ \sum_{v=m+1}^{\infty} c_{j_v}^{-1} \right\}^p. \end{aligned}$$

Hence

$$\varphi_k(f) \geq \frac{1}{2} c_{j_m}^{1-p} - 2.$$

Letting  $m \rightarrow \infty$  we obtain

$$\limsup_{k \rightarrow \infty} \varphi_k(f) = \infty.$$

The contradiction show that

$$\limsup_{k \rightarrow \infty} \sup_{\|f\|_C \leq 1} \varphi_k(f) < \infty$$

which is equivalent to (8).

## References

### Books

- Ahieser, N.I., Theory of Approximation, Fr. Ungar Publ. Co., New York, 1956.  
 Freud, G., Orthogonal Polynomials, Pergamon Press, New York, 1971.  
 Geronimus, L. Ya., Orthogonal Polynomials, Consultants Bureau, New York, 1961.  
 Grenander, U. and Szegő, G., Toeplitz Forms and Their Applications, Berkeley, Los Angeles, 1958.  
 Szegő, G., Orthogonal Polynomials, AMS, New York, 1967.

### Periodicals

- [1] Askey, R., Mean convergence of orthogonal series and Lagrange Interpolation, *Acta. Math. Acad. Sci. Hungar.* 23 (1972), 71-85.  
 [2] Badkov, V., Convergence in mean and almost everywhere of Fourier series in orthogonal polynomials, *Mat. Sbornik* 95 (137) (1974), 229-262.  
 [3] Carleson, L., On convergence and growth of partial sums of Fourier series, *Acta Math.* 116 (1966), 135-157.  
 [4] Case, K. M., Orthogonal polynomials revisited, in *Theory and Application of Special Functions*, ed. R. A. Askey, Academic Press, 1975, 289-304.  
 [5] Erdős, P. and Turán, P., On interpolation. III, *Annals of Math.* 41 (1940), 510-553.  
 [6] Freud, G., Über die Konvergenz des Hermite-Fejerschen Interpolationsverfahrens, *Acta. Math. Acad. Sci. Hungar.* 5 (1954), 109-128.



- [ 7 ] Freud, G., Über eine Klasse Lagrangescher Interpolationsverfahrens, Studia Sci. Math. Hungar. 3 (1968), 249-255.
- [ 8 ] Freud, G., On Hermite-Fejer interpolation processes, Studia Sci. Math. Hungar. 7 (1972), 307-316.
- [ 9 ] Khalilova, B., On some estimates for polynomials, Izvestija AN Azerb. SSR 2 (1974), 46-55.
- [ 10 ] Natanson, G.I., Two sided estimation of Lebesgue function of Lagrange interpolation with Jacobi nodes, Izvestija Vyssh. Uc. Zav. (Matematika) 11(66) (1967), 67-74.
- [ 11 ] Nevai, G. P., Orthogonal polynomials on the real axis with respect to the weight  $|x|^{\alpha} \exp(-|x|^{\beta})$ , I, Acta Math. Acad. Sci. Hungar. 24 (1973), 335-342.
- [ 12 ] Nevai, G. P., Mean convergence of Lagrange interpolation. I, J. of Approximation Th. 18 (1976), 363-377.

Department of Mathematics and  
Mathematics Research Center  
University of Wisconsin-Madison  
Madison, Wisconsin 53706

and

Department of Mathematics  
The Ohio State University  
Columbus, Ohio 43210.



REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1726 (14) MRC-TSR-1726	2. GOVT ACCESSION NO. (9)	3. RECIPIENT'S CATALOG NUMBER Technical
4. TITLE (and Subtitle) ORTHOGONAL POLYNOMIALS		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period
7. AUTHOR(s) Paul G. Neval		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 NSF-MPS75-06687 #3
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 6 - Spline Functions and Approximation Theory
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 Below. (12) 120p.		12. REPORT DATE February 1977
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 231
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 American Mathematical Society Providence, Rhode Island 02940 National Science Foundation Washington, D. C. 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Orthogonal polynomials, Quadrature processes, Fourier series, Interpolation, Positive operators, Toeplitz matrices, Christoffel functions.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegő and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem will (positively) be answered: is there any weight $w$ with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to $w$ diverge in $L_w^p$ for some continuous function $f$ ? Most of the paper deals with Christoffel functions and their applications. Many limit relations for orthogonal polynomials are found in the		